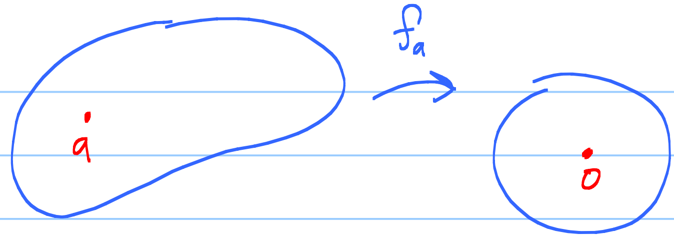
Lecture 31 Harmonic functions, Dirichlet problem



a) The map constructed in my pf of the R. map. Thm, is such that $f(a) = 0$.

Why: If $f(a) = b \neq 0$, compose with $\varphi_b(z) = \frac{z-b}{1-\bar{b}z}$, get a map with bigger derivative at a. ↯.

b) If f and $\tilde{f}$ are two Riemann maps associated to $a \in \Omega$, then $f = \lambda \tilde{f}$ for const λ with $|\lambda| = 1$. [Schwarz]

c) The R. Map associated to $a \in \Omega$ is the one with $f'(a) \in \mathbb{R}^+$.

Koebe's Proof of R.M.T.: Step 1: Cook up 1-1 analytic map into $D_1(0)$. $[\tilde{\mathcal{F}} \neq \emptyset.]$

Step 2: Say $w_0 \in D_1(0)$ gets missed. Do $\sqrt{\ } \circ$ Möbius trick to get map with a strictly bigger derivative at a.

Step 3: Repeat.

Step 4: Show that seq $\longrightarrow$ to an onto map.

Harmonic fcns: u C^2-smooth and $\Delta u \equiv 0$.

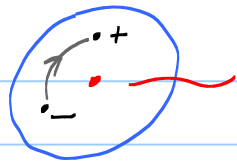
1) $u = \operatorname{Re} f$, f analytic, locally and globally on a s.c. connected domain.

2) Harmonic fcns satisfy the Averaging Property:

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})\, d\theta \quad \text{if } \overline{D_\rho(z_0)} \subset \Omega.$$

Why: $u = \mathrm{Re}\, f$, f satisfies ave prop. Take Re part.

HWK #6:



$u(z_0)=0$

or use $u = \mathrm{Re}\, f$ and the Local mapping thm.

Big Fact: A continuous fcn that satisfies the averaging property must be harmonic!

Thm: Harmonic fcns satisfy the Max Principle. And the Minimum Principle.

Why: Ave Prop $\Rightarrow$ Max Princ.

Max Princ for $-u$ $\Rightarrow$ Min Princ for u.

Remark: Say u attains a local max at z_0. $D_r(z_0) \subset \Omega$.

$u = \mathrm{Re}\, f$ on $D_r(z_0)$. $\quad e^u = |e^u \cdot e^{iv}| = |e^f|$

local max at z_0 $\qquad$ Max Princ $\Rightarrow e^f = \text{const}$,

So $e^u \equiv c$, and $u \equiv \mathrm{Ln}\, c$ on $D_r(z_0)$. $\Rightarrow u \equiv \mathrm{Ln}\, c$ on Ω.

Dirichlet Problem on $D_1(0)$: Given a continuous fcn on $C_1(0)$: $\varphi(e^{i\theta})$, find a fcn u continuous on $\overline{D_1(0)}$ and harmonic on $D_1(0)$ with $u(e^{i\theta}) = \varphi(e^{i\theta})$.

Step 1: Easy if $\varphi(x,y)$ is a polynomial.

$$\varphi(x,y) = \varphi\left(\frac{z+\bar z}{2}, \frac{z-\bar z}{2i}\right) = \sum_{n,m=0}^{N} c_{nm}\, z^n \bar z^m$$

Key: $z = \frac{1}{\bar z}$ on circle. $\left[\, e^{i\theta} = \frac{1}{e^{-i\theta}} \,\right]$

$\bar{z} = \frac{1}{z}$ on circle too

Big idea: $z^n \bar{z}^m$ $\quad$ $m > n$: $= \underbrace{z^n \bar{z}^n}_{\substack{=1 \\ \text{on } C_1(0)}} \underbrace{\bar{z}^{m-n}}_{\text{harmonic}}$

Aha! $\bar{z}^{m-n}$ solves D. Prob for $z^n \bar{z}^m$.

$m < n$: $z^n \bar{z}^m = z^{n-m} \underbrace{z^m \bar{z}^m}_{=1 \text{ on } C_1(0)}$ So z^{n-m} solves D. Prob for $z^n \bar{z}^m$

$m = n$: $z^n \bar{z}^n = 1$ on $C_1(0)$. So 1 solves D. Prob for $z^n \bar{z}^n$.

Fact: Given polynomial data, solution to D. Prob. is poly.

Thm: Max Princ $\Longrightarrow$ Sol^n to D. Prob is unique.

Why: u_1 and u_2 solve D. Prob for φ, $u_1 - u_2$ is harmonic and $= 0$ on $C_1(0)$. Max Princ #2 $\Rightarrow$ Max on bndry. So $u_1 - u_2 \le 0$ on $\overline{D_1(0)}$. Repeat for $u_2 - u_1$ to see $u_2 - u_1 \le 0$ [or use Min Princ for $u_1 - u_2$]. So $u_1 \equiv u_2$ on $D_1(0)$.

Plan to solve D. Prob: Step 1: Find polynomials $\varphi_n(x,y)$ converging uniformly to given φ on $C_1(0)$.

Step 2: Get polynomial sol^{n}s $\quad p_n(z) + q_n(\bar{z}) = u_n(z)$ to D. Prob for φ_n.

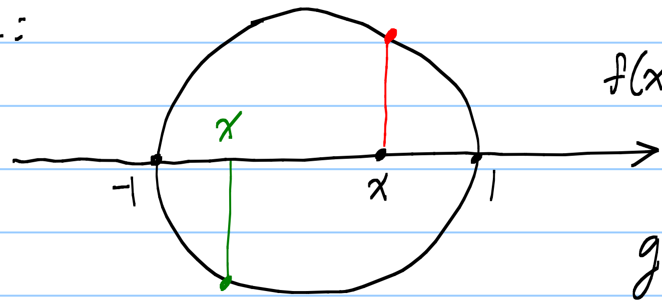
Step 3: Max Princ $\Rightarrow$ u_n conv uniformly on $\overline{D_1(0)}$.

$u_n \to u$

Step 4: $\quad u(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left[\frac{e^{i\theta} + z}{e^{i\theta} - z}\right] u(e^{i\theta})\, d\theta$

$= \mathrm{Re}\,[\text{Analytic fcn}]$. So u is harmonic.

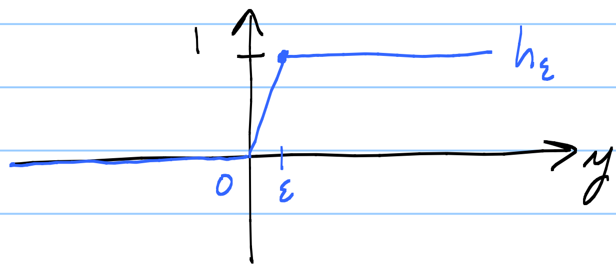
Pf of Step 1:



$f(x) = \varphi(x, \sqrt{1-x^2})$ top

$g(x) = \varphi(x, -\sqrt{1-x^2})$ bottom

Can subract off a linear fcn $Ax+B$ to reduce to case $\varphi(-1)=0$ and $\varphi(1)=0$.



Weierstraß Thm: Get polys $f_n \to f$ on $[-1,1]$

$g_n \to g$ on $[-1,1]$

$h_n \to h_\varepsilon$ on $[-1,1]$.

Finally $f_n(x)\,h_n(y) + g_n(x)\,h_n(-y)$ is a poly on $\mathbb{R}^2$ close to given $\varphi(e^{i\theta})$ on $C_1(0)$.