Lecture 36 General Cauchy Theorem

Thm: Ω a domain, Γ a cycle in Ω, f analytic on Ω, and $\mathrm{Ind}_\Gamma(a) = 0$ for all $a \in \mathbb{C} - \Omega$. Then

1) $\displaystyle\int_\Gamma f\,dz = 0$

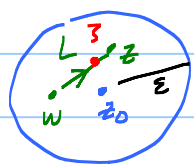
2) $\displaystyle\mathrm{Ind}_\Gamma(a)\,f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz, \quad a \in \Omega - tr(\Gamma).$

Pf: (Ahlfors) Let

$$g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases}$$

Step 1: g is continuous on $\Omega \times \Omega$.

Clear near (z_0, w_0) when $z_0 \neq w_0$. Only need to check near (z_0, z_0).



$$g(z,w) - g(z_0,z_0) =$$

$$\frac{f(z)-f(w)}{z-w} - f'(z_0) =$$

$$\frac{1}{z-w}\int_L f'(\zeta)\,d\zeta - f'(z_0)\underbrace{\left(\frac{\int_L 1\,d\zeta}{z-w}\right)}_{=1}$$

$$= \frac{1}{z-w}\int_L \left(f'(\zeta) - f'(z_0)\right)d\zeta$$

So $\displaystyle |g(z,w) - g(z_0,z_0)| \leq \frac{1}{|z-w|}\left(\max_{D_\varepsilon(z_0)} |f'(\zeta) - f'(z_0)|\right)\cdot|z-w|$

can be made small by taking ε small since f' is continuous.

Step 2: Define $\displaystyle h(z) = \frac{1}{2\pi i}\int_\Gamma g(z,w)\,dw$

Plan: Extend h to $\mathbb{C}$ as an entire fcn!

Easy to see $h(z) \to 0$ as $z \to \infty$. So h bnd, entire.

Liouville's $\Rightarrow$ $h \equiv c$, which must be zero.

$$0 = h(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)-f(w)}{z-w}\, dw \quad \leftarrow z \notin tr(\Gamma),\ z \neq w$$

$$= f(z) \cdot \underbrace{\frac{1}{2\pi i} \int_\Gamma \frac{dw}{z-w}}_{-\mathrm{Ind}_\Gamma(z)} - \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{z-w}\, dw \quad \checkmark$$

Step 3: h is analytic on Ω.

1) Clear that h is continuous on Ω.

[$g(z,w)$ cont. on $\Omega \times \Omega$, $tr(\Gamma)$ is compact, unif continuity on compact set $\overline{D_\varepsilon(z)} \times tr(\Gamma)$.]

2) $Q \subset \Omega$, Q a triangle.

$$\int_Q h\, dz = \int_Q \left(\int_\Gamma g(z,w)\, dw \right) dz$$

$$\underset{\substack{\uparrow \\ \text{Fubini's Thm}}}{=} \int_\Gamma \left(\underbrace{\int_Q g(z,w)\, dz}_{} \right) dw$$

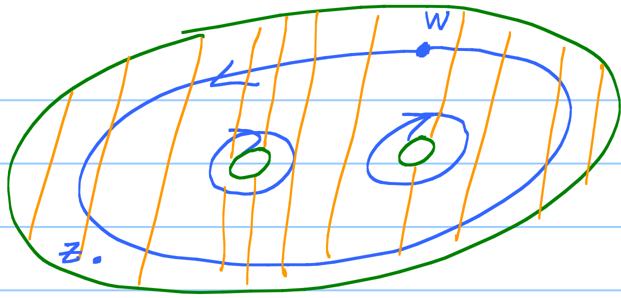
$$= \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases}$$

Aha! If w is fixed, $g(z,w)$ has a removed sing. at $z=w$,

So $\int_Q g\, dz = 0$ by Goursat.

Conclude that $\int_Q h\, dz = 0$. Morera's $\Rightarrow$ h analytic.

Step 4: Extend h to $\mathbb{C}$



$\tilde{\Omega} = \{ z \in \mathbb{C} - tr(\Gamma) : Ind_\Gamma(z) = 0 \}$

Note: 1) $\mathbb{C} - \Omega \subset \tilde{\Omega}$ by Hyp.

2) $\tilde{\Omega}$ is open

3) $\mathbb{C} = \Omega \cup \tilde{\Omega}$.

Hmmm.

$$h(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z) - f(w)}{z - w}\, dw$$

$$= \underbrace{\frac{1}{2\pi i} f(z) \int_\Gamma \frac{dw}{z-w}}_{\text{Aha! } = 0 \text{ in } \Omega \cap \tilde{\Omega} \text{ and in } \mathbb{C} - \Omega.} - \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{z-w}\, dw$$

Define $\tilde{h}(z) = -\frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{z-w}\, dw$ on $\tilde{\Omega}$.

Claim: $H(z) = \begin{cases} h(z) & z \in \Omega \\ \tilde{h}(z) & z \in \tilde{\Omega} - \Omega \end{cases}$ is entire!

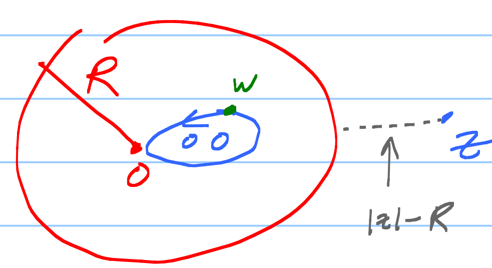
Pick $z_0 \in \mathbb{C}$. Cases: $z_0 \in \Omega$ ✓

$z_0 \in \tilde{\Omega}$ Diff under $\int$ in z.

$z_0 \in \Omega \cap \tilde{\Omega}$, same thing.

Step 5: $H(z) \to 0$ as $z \to \infty$.

$tr(\Gamma) \subset D_R(0)$



Clear that $Ind_\Gamma(z) = 0$ for

z outside $D_R(0)$.

So $\quad H(z) = -\frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{z-w}\,dw \quad$ there

$$|H(z)| \le \frac{1}{2\pi}\left(\max_{tr(\Gamma)} |f|\right)\cdot \frac{1}{|z|-R}\cdot \text{Length}(\Gamma)$$

$$\longrightarrow 0 \quad \text{as} \quad z\to\infty.$$

Step 6: H bnd entire, so const. Since $\to 0$ as $z\to\infty$, the const. is zero. Done!

Step 7: Standard trick to get Cauchy Thm from the Integral Formula: Pick $z_0 \in \Omega - tr(\Gamma)$.

Given analytic f on Ω, let $F(z) = (z-z_0)f(z)$.

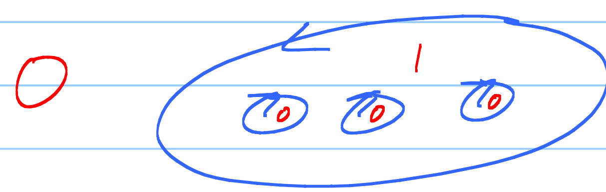
Apply Cauchy Int. Formula to $F(z)$ at z_0 =

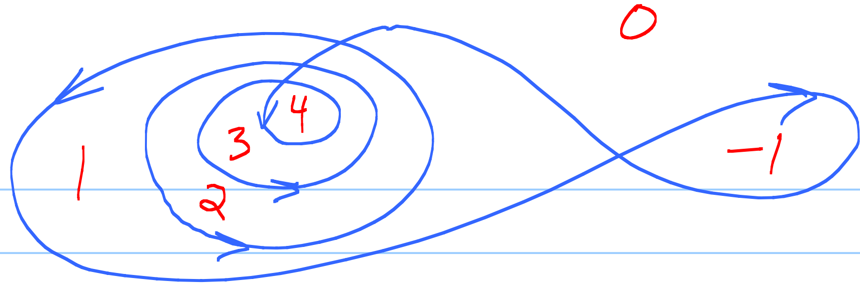
$$\frac{1}{2\pi i}\int_\Gamma \underbrace{\frac{F(w)}{w-z_0}}_{\frac{(w-z_0)f(w)}{w-z_0} = f(w)}\,dw = \underbrace{F(z_0)}_{=0}\,\mathrm{Ind}_\Gamma(z_0) = 0$$

Get $\int_\Gamma f\,dw = 0$. ✓

Def^n: Γ is a cycle if $\Gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$

where γ_j's are closed curves.





Fact: $\mathrm{Ind}_\Gamma(z)$ is constant on connected components of $\mathbb{C} - \mathrm{tr}(\Gamma)$.

Why:
$$\frac{d}{dz}\left(\mathrm{Ind}_\Gamma(z)\right) = \frac{1}{2\pi i}\sum \int_{\gamma_j} \frac{1}{(w-z)^2}\,dw$$
$$= \frac{1}{2\pi i}\sum \int_{\gamma_j} \frac{d}{dw}\left[\frac{-1}{w-z}\right] dw = \sum 0$$

because γ_j's closed

Fact: $\mathrm{Ind}_\Gamma(z) \equiv 0$ on unbounded component of $\mathbb{C} - \mathrm{tr}(\Gamma)$.

Because basic estimate shows $\mathrm{Ind}_\Gamma(z) \to 0$ as $z \to \infty$,

Def^n: Simple cycle: