Lecture 37 General Residue Theorem

General Cauchy Thm: Ω a domain, f analytic on Ω, Γ a cycle in Ω and $\underbrace{\mathrm{Ind}_\Gamma(a) = 0 \text{ for all } a \in \mathbb{C} - \Omega}_{\Gamma \sim 0}$. Then

1) $\int_\Gamma f \, dz = 0$

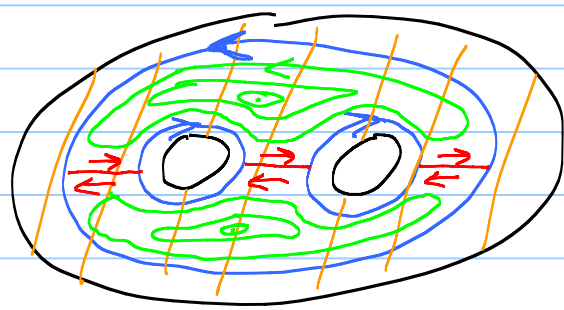
2) $\mathrm{Ind}_\Gamma(z_0) f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} dz$, $z_0 \in \Omega - tr(\Gamma)$.

Def^n: Condition is called "Γ is homologous to zero."

Write $\Gamma \sim 0$ or $\Gamma \sim_\Omega 0$.

How to feel about it:

Add cuts to get closed curves homotopic to pts in Ω.



Remark: $\Gamma \sim 0$ for any cycle in a simply connected domain.

$a \notin \Omega$, $\int_\gamma \underbrace{\frac{1}{z-a}}_{\text{analytic on s.c. } \Omega} dz = 0$ by Cauchy Thm.

Theorem: New improved Cauchy Thm for Ω a simply connected domain, f analytic on Ω, γ closed.

1) $\int_\gamma f \, dz = 0$ ✓

2) $\mathrm{Ind}_\gamma(z_0) f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$ $\leftarrow$ New

General Residue Thm: Ω a domain, Γ cycle ~ 0 in Ω, f analytic on Ω except at finitely many isolated sing. $A = \{a_1, \dots, a_N\}$ in Ω. Suppose $A \cap tr(\Gamma) = \emptyset$.

Then
$$\int_\Gamma f\,dz = 2\pi i \sum_{n=1}^{N} \mathrm{Ind}_\Gamma(a_n)\, \mathrm{Res}_{a_n} f$$

Pf: $\tilde{\Omega} = \Omega - A$. Let $n_j = \mathrm{Ind}_\Gamma(a_j)$.

Can find $\varepsilon > 0$ so small that $\overline{D_\varepsilon(a_i)} \cap \overline{D_\varepsilon(a_j)} \cap tr(\Gamma) = \emptyset$ for $i \neq j$.

$C_j : z(t) = a_j + \varepsilon e^{it}, \quad 0 \le t \le 2\pi$

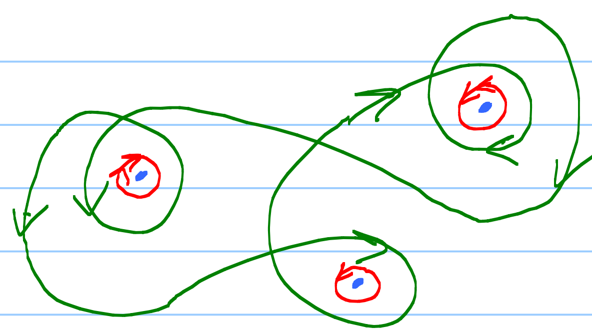
$mC_j : z(t) = a_j + \varepsilon e^{imt}, \quad 0 \le t \le 2\pi$

goes around m times.

Note $\int_{mC_j} f\,dz = m\int_{C_j} f\,dz = m\,2\pi i\,\mathrm{Res}_{a_j} f$

Aha! $\tilde{\Gamma} = \Gamma \cup \left(\bigcup_{j=1}^{N} -n_j C_j \right)$

Now $\tilde{\Gamma} \sim 0$ in $\tilde{\Omega}$.



f analytic on $\tilde{\Omega}$.
General Cauchy $\Rightarrow$

$$\int_{\tilde{\Gamma}} f\,dz = 0$$

$$\int_\Gamma f\,dz + \sum \int_{-n_j C_j} f\,dz = 0$$

$$-n_j \int_{C_j} f\,dz = \left(-\mathrm{Ind}_\Gamma(a_j)\right) 2\pi i \,\mathrm{Res}_{a_j} f \quad \checkmark$$

<u>Def</u>^{n}: f is called meromorphic on a domain Ω if f is analytic on $\Omega - A$ where A is a discrete subset of Ω where f has poles.

<u>Note</u>: Discrete sets have no limit pts.

<u>Remark</u>: General Residue Thm extends to meromorphic f:

$$\int_\Gamma f\,dz = 2\pi i \sum_{a\in A} \mathrm{Ind}_\Gamma(a)\,\mathrm{Res}_a f$$

where the sum is <u>finite</u> because $\mathrm{Ind}_\Gamma(a) = 0$ for all but finitely many a's. [Because

$S = \{a \in \mathbb{C} : \mathrm{Ind}_\Gamma(a) \neq 0\} \cup tr(\Gamma)$ is compact, $\subset \Omega$.

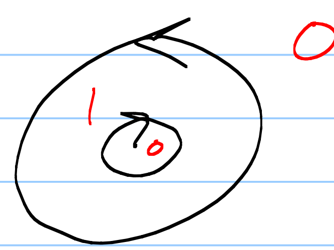
$A \cap S$ has to be finite to not have limit pts in Ω.]

<u>Def</u>^{n}: Γ "simple cycle": $\mathrm{Ind}_\Gamma(a) = 0$ or $\mathrm{Ind}_\Gamma(a) = 1$

Think: "Inside of Γ"

$= \{z : \mathrm{Ind}_\Gamma(z) = 1\}$

"Outside" $= \ldots\ldots$ 0



General Thms nicer for <u>simple</u> cycles.

<u>Cauchy</u>: f is analytic inside and on on Γ.

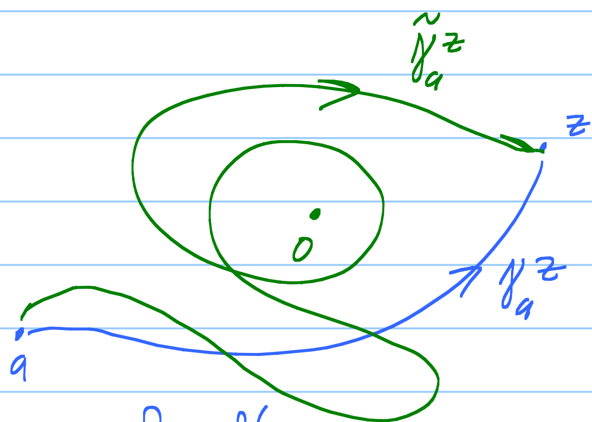
Then $\int_\Gamma f\,dz = 0$.

<u>Residue</u>: $$\int_\Gamma f\,dz = 2\pi i \sum_{a\in A} \mathrm{Res}_a f$$ where A = sing. of f inside Γ.

<u>Argument Princ</u>: $$\frac{1}{2\pi i}\int_\Gamma \frac{f'}{f}\,dz = \left(\begin{array}{l}\#\text{ zeroes of } f\\ \text{inside } \Gamma\\ \text{with mult}\end{array}\right)$$

<u>or</u> $= (\#\text{zeroes}) - (\#\text{poles})$ if poles allowed.

<u>Hwk 9: 1.</u>:

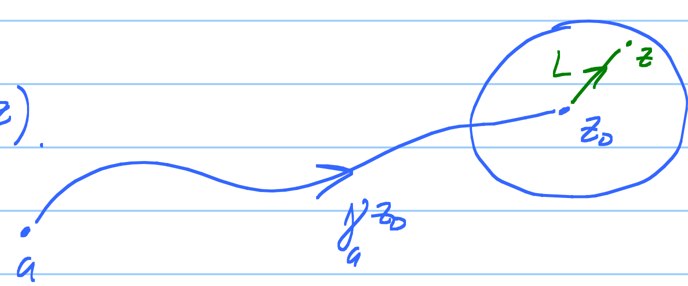


$\gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$

$$F(z) = e^{\frac{1}{3}\int_{\gamma_a^z} \frac{f'}{f}\,dw}$$

$$e^{\frac{1}{3}\underbrace{\left(\int_{\gamma_a^z} + \int_{-\tilde{\gamma}_a^z}\right)\frac{f'}{f}\,dw}_{2\pi i\,\mathrm{Ind}_\gamma(0)\,\mathrm{Res}_0 \frac{f'}{f}}} = \frac{e^{\frac{1}{3}\int_{\gamma_a^z}}}{e^{\frac{1}{3}\int_{\tilde{\gamma}_a^z}}} \quad \text{want} = 1$$

<u>Note</u>: Compute $F'(z)$.

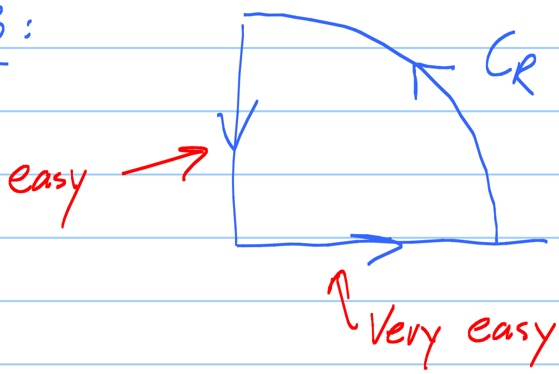


F' satisfies $\quad F' = \frac{1}{3}\frac{f'}{f}F.$

Wish $\quad F^3 = f.$

Consider $\quad \frac{F^3}{f} \quad$ and $\quad \frac{d}{dz}\left(\frac{F^3}{f}\right).$

Prob 3:



Increase in Arg P

$$\underbrace{\int_{C_R} \frac{f'}{f}\,dz}_{\xrightarrow{R\to\infty}} = \begin{pmatrix}\text{real}\\ \text{part}\end{pmatrix} + i\,\Delta_{C_R} \arg f$$

don't care about this around the whole closed loop