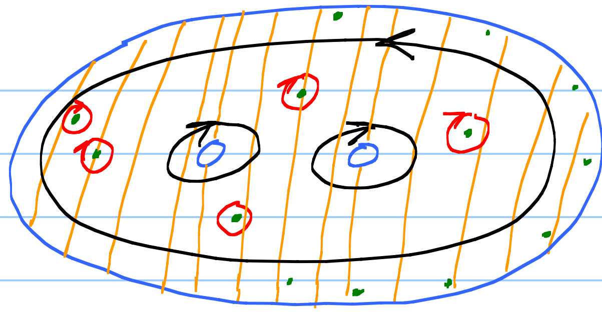
Lecture 38 Mittag-Leffler Theorem & Weierstraß Thm

General Residue Thm



A = discrete set of singular points in Ω

$A \cap tr(\Gamma) = \emptyset$

Mittag-Leffler Thm: Given a seq of distinct points $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ such that $a_n \to \infty$ as $n \to \infty$ (= no limit pts in $\mathbb{C}$). Given principal parts

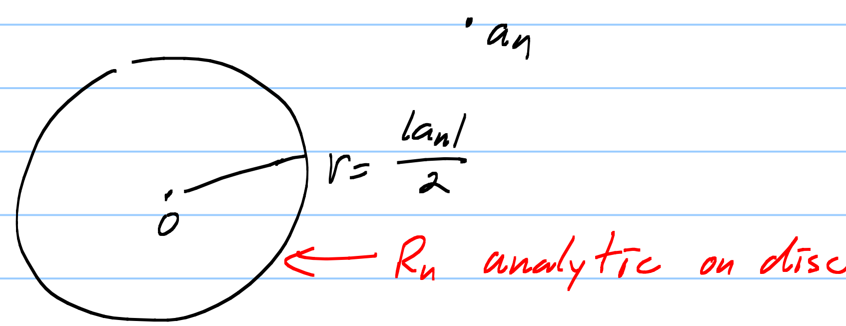
$$R_n(z) = \frac{A_{N_n,n}}{(z-a_n)^{N_n}} + \cdots + \frac{A_{1,n}}{(z-a_n)},$$

there exists an analytic fcn f on $\mathbb{C} - \{a_n\}_{n=1}^{\infty}$ with poles at each a_n with p.p. R_n [$f - R_n$ has a removable sing at a_n for each n].

Note: f is meromorphic on $\mathbb{C}$.

Pf: First try $f = \sum_{n=1}^{\infty} R_n(z)$.

Idea: $r_n = \frac{|a_n|}{2}$



Approx R_n on $D_{r_n}(0)$ by a Taylor poly $p_n(z)$ with

$$|R_n(z) - p_n(z)| < \frac{1}{2^n}$$

Second try: $f(z) = \sum_{n=1}^{\infty} \left(R_n(z) - p_n(z) \right)$

Claim: f solves the M-L Prob.

Consider f on $D_R(0)$. Since $a_n \to \infty$ as $n \to \infty$, $\exists N$ such that $\frac{|a_n|}{2} > R$ if $n > N$.

$$f(z) = \underbrace{\sum_{n=1}^{N} \left(R_n(z) - p_n(z) \right)}_{\substack{\text{Rational fcn with} \\ \text{correct prescribed poles} \\ \text{inside } D_R(0)}} + \sum_{n=N+1}^{\infty} \underbrace{\left[R_n(z) - p_n(z) \right]}_{\substack{\text{analytic on } D_R(0) \\ \text{and } |R_n - p_n| < \frac{1}{2^n} \\ \text{there.}}}$$

Weierstraß M-Test $\Rightarrow \Sigma$ converges unif on $D_R(0)$ to an analytic fcn there.

Near a_n: $\quad f(z) = R_n(z) + (\text{analytic near } a_n)$ ✓

Remark: M-L Thm is true on any domain in $\mathbb{C}$.

Weierstraß Thm: Given a discrete set of pts $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ [$a_n \to \infty$ as $n \to \infty$] and positive integers m_n, there is an entire fcn having zeroes only at the points a_n with prescibed multiplicity m_n.

Remark: Classic proof uses infinite products.

My proof: (M-L + General Residue Thm $\Rightarrow$ Weierstraß.)

Step 1: M-L $\Rightarrow$ there is a meromorphic fcn $F(z)$ on $\mathbb{C}$ with simple poles at each a_n with residue m_n and no other poles. [p.p. at a_n is $\dfrac{m_n}{z-a_n}$]

Think: $\quad f = e^{\log f} = e^{\int_{\gamma_a^z} \frac{f'}{f}\, dz} \underset{\text{near } a_n}{=} e^{\int_{\gamma_a^z} \underbrace{\frac{m_n}{w-a_n} + (\text{analytic})}_{\text{use } F(z) \text{ here!}} dw}$

Idea: Try $\quad f(z) = \exp\left(\int_{\gamma_a^z} F(w)\, dw \right)$ where γ_a^z is a curve from a fixed pt a going to z, avoiding a_n's.

Step 1: f is well defined.

$$\frac{f(z)}{\tilde f(z)} = \exp\Big(\underbrace{\Big(\int_{\gamma_a^z} - \int_{\tilde\gamma_a^z}\Big)}_{\int_\Gamma} F(w)\,dw \Big)$$

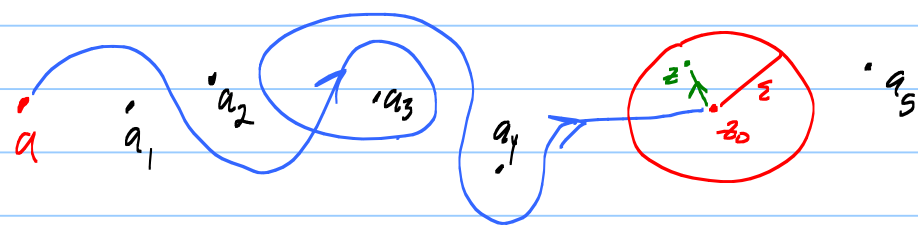
$$= \exp\Big(2\pi i \sum_{n=1}^{\infty} \underbrace{\mathrm{Ind}_\Gamma(a_n)}_{\in \mathbb{Z}} \underbrace{\mathrm{Res}_{a_n} F}_{m_n \in \mathbb{Z}^+} \Big) = 1$$

(finite sum) (Gen. Res. Thm. $\Omega = \mathbb{C}$)

So $f(z) = \tilde f(z)$. ✓

Step 2: f is analytic on $\mathbb{C} - \{a_n\}_{n=1}^{\infty}$.

Restrict attention to $D_\varepsilon(z_0) \subset \mathbb{C} - \{a_n\}_1^\infty$.

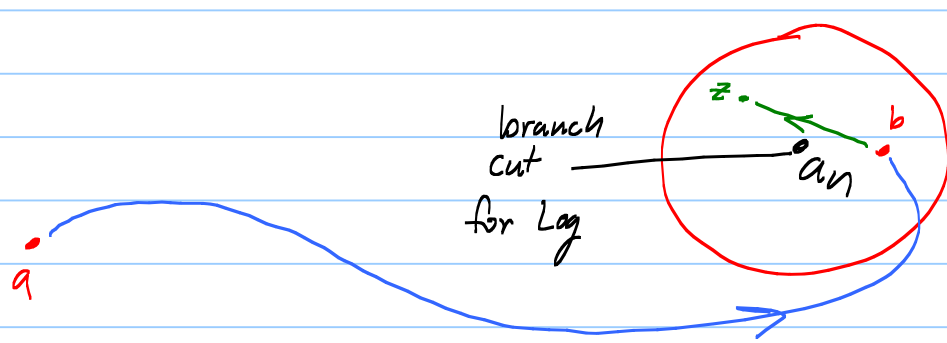


$$\gamma_a^z = \gamma_a^{z_0} \cup L_{z_0}^z$$

on $D_\varepsilon(z_0)$:
$$f(z) = \exp\Big(\underset{\uparrow\ \int_{\gamma_a^{z_0}}}{c} + \underbrace{\int_{L_{z_0}^z} F(w)\,dw}_{\text{analytic, Deriv. } = F(z)} \Big)$$

See f analytic and $f'(z) = f(z)F(z)$.

Step 3: a_n is a removable sing for $f(z)$:



$$f(z) = \exp\Big(\underset{\uparrow \atop \int_{\gamma_a^b} F\,dw}{c} + \underbrace{\int_{L_b^z} F \, dw}_{\underbrace{\int_{L_b^z} \frac{m_n}{z-a_n}}_{m_n \operatorname{Log}(w-a_n)\big|_b^z} + H(w)\,dw} \Big)$$

$H(w)$ ← analytic near a_n

$$= \underbrace{\exp\big(m_n \operatorname{Log}(z-a_n)\big)}_{(z-a_n)^{m_n}} \cdot \underbrace{\exp\big(\text{Nice and analytic}\big)}_{h(z)}$$

$h(z)$ analytic near a_n and $h(a_n) \neq 0$.

Step 4: Aha! f is analytic on $D_\varepsilon(a_n) - (\text{branch cut})$, continuous up to the cut. So it extends to be analytic on $D_\varepsilon(a_n) - \{a_n\}$. And a_n is removable, in fact a zero of order m_n. Done!

Weierstraß: $\prod_{n=1}^{\infty} (z - a_n)$ No!

$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)$ Works for $m_n = 1$, $a_n = n^2$

Big Idea: Weierstraß polys to multiply by to make terms $\to 1$ fast enough.

Practice problems on Home Page.