Lecture 39 Zeroes & poles

Cor: (Weierstraß $\Rightarrow$) $\mathcal{M}$ = meromorphic fcns on $\mathbb{C}$

$$\mathcal{M} = \left\{ \frac{f}{g} : f, g \text{ entire}, g \not\equiv 0 \right\}.$$

Pf: Suppose H is mero on $\mathbb{C}$. Then $\exists$ seq $\{a_n\}_{n=1}^{\infty}$ where H has poles, say of order m_n at each a_n. Weierstraß $\Rightarrow$ $\exists$ g entire with zeroes of order m_n at a_n and no other zeroes. Now $gH = f$ has removable sing at each, so is entire. ✓

Cor: (M-L + W $\Rightarrow$) Can cook entire fcns with specified first m_n terms at a_n at a discrete set of pts $\{a_n\} \subset \mathbb{C}$.

Remark: If f analytic on $D_\varepsilon(\infty) = \{z : |z| > \frac{1}{\varepsilon}\}$. Then f has an "isolated singularity at ∞." The type of the sing is defined to be the type of the sing of $f(\frac{1}{z})$ at $z=0$.

Facts: 1) An entire fcn with a pole at ∞ $\left[\lim_{z\to\infty} f(z) = \infty\right]$ is a polynomial

2) An entire fcn with a removable sing is constant.

3) Meromorphic fcns on $\hat{\mathbb{C}}$ = Rational fcns.

$\left[\begin{array}{l} f \text{ with finitely many poles in } \mathbb{C} \\ \text{and pole or removable sing at } \infty \end{array}\right]$ is rational.

4) Non poly entire fcns have essential sing at ∞.

Infinite products: $\prod_{n=1}^{\infty} p_n = \lim_{N\to\infty} \prod_{n=1}^{N} p_n$

where all $p_n \in \mathbb{C}$, $p_n \neq 0$.

Lemma: If $\prod_{n=1}^{\infty} p_n$ exists and $= L \neq 0$, then $p_n \to 1$ as $n \to \infty$.

Pf: $$p_N = \frac{\prod_1^N p_n}{\prod_1^{N-1} p_n} \longrightarrow \frac{L}{L} = 1 \quad \text{as } N\to\infty.$$

Traditional to write: $\prod_{n=1}^{\infty} (\underbrace{1+a_n}_{p_n})$, $a_n \to 0$.

Lemma: $\prod_{n=1}^{\infty} (1+a_n)$ converges to $L \neq 0$ $\Longleftrightarrow$ $\sum_{n=1}^{\infty} \operatorname{Log}(1+a_n)$ converges.

Pf: $\Leftarrow$ easy. $e^{\sum^N \operatorname{Log}(1+a_n)} = \prod^N (1+a_n)$

$\Rightarrow$ Read in Stein or Ahlfors.

Fact: $$(1-\varepsilon)|a_n| \leq |\operatorname{Log}(1+a_n)| \leq (1+\varepsilon)|a_n| \quad (*)$$

for a_n close to zero because $\operatorname{Log}(1+z)$ has a simple zero at $z=0$ and $\left.\frac{d}{dz}\operatorname{Log}(1+z)\right|_{z=0} = 1$.

Def^n: An infinite product of non-zero terms $\prod_1^{\infty}(1+a_n)$ **converges absolutely** if $\sum_1^N a_n$ does.

Lemma: $\sum_{n=1}^{\infty} \operatorname{Log}(1+a_n)$ converges absolutely exactly when $\sum_1^{\infty} a_n$ converges absolutely [because of $(*)$]

exactly when $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely.

Fact: If $\sum_{n=1}^{\infty} \underbrace{f_n(z)}_{\text{analytic}}$ converges absolutely and uniformly on compact subsets of a domain Ω, then $\prod_{n=1}^{\infty}(1+f_n(z))$ converges absolutely and unif on compact subsets to an analytic fcn.

EX: $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = f(z)$

$$\sum \left|\frac{z^2}{n^2}\right| = |z|^2 \underbrace{\sum \frac{1}{n^2}}_{\frac{\pi^2}{6}} < \infty$$

So f is entire. And f has simple zeroes at $z \in \mathbb{Z}$.

[Factor out $\left(1-\frac{z^2}{n^2}\right)$ term near $n \in \mathbb{Z}$.]

EX: $\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)$ Ouch. $\sum \frac{1}{n} = \infty$

Weierstraß: $\prod_{n=1}^{\infty} \underbrace{\left(1-\frac{z}{n}\right) e^{-\mathrm{Log}\left(1-\frac{z}{n}\right)}}_{=1}$

$$-\mathrm{Log}(1-z) = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots$$

$$P_N(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^N}{N}$$

$(1-z)e^{P_N(z)}$ as close to 1 as needed on $D_{1/2}(0)$ by taking N large.

Famous formulas,

$$\sin \pi z = \pi z \prod_{n \neq 0}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}$$

$$= \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

Step 1: $\sum \frac{1}{(z-n)^2}$ converges to a mero fcn with double poles at $\mathbb{Z}$.

Step 2: $\frac{\pi^2}{\sin^2 \pi z}$ has same poles with same Princ. parts.

Step 3: Both are periodic with period 1.

Step 4: Show that diff is bndd on $\{z: 0 \le \operatorname{Re} z \le 1\}$.

Step 5: Liouville's.