Lecture 40 Radó's Theorem

Theorem: Suppose u is continuous real valued fcn on a domain Ω such that u_x, u_y, u_{xx}, u_{yy} exist and $u_{xx}+u_{yy}\equiv 0$ on Ω. Then u is C^∞-smooth and harmonic [so locally the real part of an analytic fcn].

Pf: May assume $\Omega = D_1(0)$ and u cont. on $\overline{D_1(0)}$.

Define $\tilde{u}(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,e^{i\theta})\, u(e^{i\theta})\, d\theta$. Want $u \equiv \tilde{u}$.

Suppose not. Then $\exists\ z_0 \in D_1(0)$ where $u-\tilde{u} \neq 0$. Assume >0 there. [Otherwise take $-u$ in place of u.]

Say $u(z_0) - \tilde{u}(z_0) = K > 0$. Pick ε with $0<\varepsilon<K$.

Let $v_\varepsilon = u - \tilde{u} + \varepsilon |z|^2$.

Note: $\Delta |z|^2 = \Delta(x^2+y^2) = 2+2 = 4$.

So $\Delta v_\varepsilon = 4\varepsilon > 0$.

Notice that $v_\varepsilon = 0 + \varepsilon$ when $|z|=1$, and

$$v_\varepsilon(z_0) = K + \varepsilon|z_0|^2 \geq K > \varepsilon$$

Hence v_ε attains its positive max on $\overline{D_1(0)}$ at a point $w_0 \in D_1(0)$.

Freshman Calculus: If $h:(a,b)\to \mathbb{R}$ is twice diff'ble and $h''(t_0)>0$, $h'(t_0)=0$, then h has a strict min at t_0.

Hence $\quad \frac{\partial^2 v_\varepsilon}{\partial x^2}(w_0) \leq 0$

$$\frac{\partial^2 v_\varepsilon}{\partial y^2}(w_0) \le 0$$

Conclude that $\Delta v_\varepsilon(w_0) \le 0$, not $4\varepsilon > 0$ ↯

Hence, $u - \tilde{u} \le 0$ on $\overline{D_1(0)}$. Repeat with $\tilde{u} - u$ in place of $u - \tilde{u}$. Get $u \equiv \tilde{u}$. ← harmonic in strong sense.

Radó's Theorem: Suppose f is a continuous complex valued fcn on a domain Ω that is analytic where it is not zero. Then f is analytic on Ω!

Pf: May assume $\Omega = D_1(0)$ with continuity up to $\overline{D_1(0)}$.

Can also divide by a constant $c \neq 0$ so that $|f| < 1$ on $\overline{D_1(0)}$. Write $u = \operatorname{Re} f$. Let $\tilde{u}(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta})\, u(e^{i\theta})\, d\theta$.

Let $v_\varepsilon = u - \tilde{u} + \varepsilon \operatorname{Ln}|f|$, $(\varepsilon > 0)$.

Note that $v_\varepsilon(e^{i\theta}) = 0 + \varepsilon \underbrace{\operatorname{Ln}|f(e^{i\theta})|}_{<0} < 0.$

Set $\operatorname{Ln}|f(z)| = -\infty$ where $f(z) = 0$.

Claim: Max Princ $\Rightarrow$ $v_\varepsilon \le 0$ on $\overline{D_1(0)}$.

Assume for now. Let $\varepsilon \searrow 0$. Get $u - \tilde{u} \le 0$.

Repeat with $\tilde{u}$ and u switched. Conclude that $u \equiv \tilde{u}$.

Start over with $v = \operatorname{Im} f$ in place of u. Get $v \equiv \tilde{v}$.

Now we have $f = u + iv$ where u and v are harmonic.

f analytic where $f \neq 0$: $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ C-R Eqns hold on open set where $f \neq 0$.

Aha! u_x, v_y, u_y, v_x are all harmonic too!

[Because $\Delta u_x = \frac{\partial}{\partial x}(\underbrace{\Delta u}_{\text{harmonic}}) \equiv 0$, etc.]

Finally, use fact that two harmonic fcns that agree on a small disc must agree on the whole domain. ✓

Pf of claim: Let $\mathcal{O}$ be a connected component of the open set in $D_1(0)$ where $f \neq 0$. Claim: $v_\varepsilon \leq 0$ in $\mathcal{O}$.

Suppose not. Then $\exists\, z_0 \in \mathcal{O}$ where $v_\varepsilon(z_0) = M > 0$, $M = \max$ on $\mathcal{O}$ since $v_\varepsilon(z) \to$ $-\infty$ or value < 0 on bndry as $z \to$ boundary pt of $\mathcal{O}$.

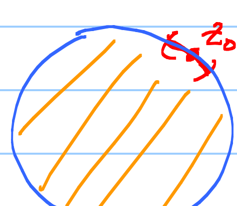
v_ε is harmonic on $\mathcal{O}$. Max Princ $\Rightarrow$ $v_\varepsilon \equiv$ const on $\mathcal{O}$.

Constant must be negative because $v_\varepsilon \to$ negative near boundary of $\mathcal{O}$. ↯.

Conclude that $v_\varepsilon \leq 0$ on $\overline{D_1(0)}$.

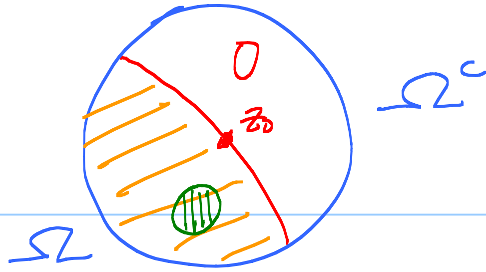
Cor: Ω domain, f analytic on Ω, f continuous on $\overline{\Omega}$, z_0 a boundary point of Ω that is not an isolated boundary pt. If f vanishes on $b\Omega \cap D_\varepsilon(z_0)$, then $f \equiv 0$ on Ω.

Picture:



f zero on red $\Rightarrow f \equiv 0$

Pf:

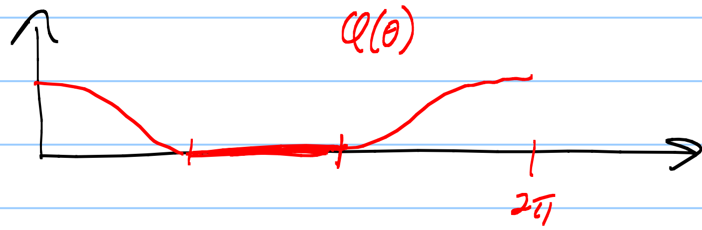


Define $\tilde{f}$ via

$$\tilde{f}(z) = \begin{cases} f(z) & z \in D_\varepsilon(z_0) \cap \overline{\Omega} \\ 0 & z \in \Omega^c \cap D_\varepsilon(z_0) \end{cases}$$

Radó's $\Rightarrow$ $\tilde{f}$ analytic. z_0 not isolated. So zero set has a limit pt. Hence $\tilde{f} \equiv 0$. But $f \equiv \tilde{f}$ on an open disc in $D_\varepsilon(z_0) \cap \Omega$. So $f \equiv 0$ on disc and hence $\equiv 0$ on Ω.

Remark: Is this true for harmonic fcns too? No!



$$u = \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta})\, \varphi(\theta)\, d\theta$$ vanishes on arc, but $\not\equiv 0$.