Lecture 41 Inhomogeneous Cauchy-Riemann Equations

The $\bar\partial$-operator:

$$dz = dx + i\,dy \qquad dx = \tfrac{1}{2}(dz + d\bar z)$$
$$d\bar z = dx - i\,dy \qquad dy = \tfrac{1}{2i}(dz - d\bar z)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$
$$= \underbrace{\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\right)}_{\frac{\partial f}{\partial z}} dz + \underbrace{\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\right)}_{\frac{\partial f}{\partial \bar z}} d\bar z$$

Def^{n}'s

Facts: $f = u + iv$. u and v satisfy the CR Equns $\Longleftrightarrow \frac{\partial f}{\partial \bar z} \equiv 0$.

If f is analytic, then $f'(z) = \frac{\partial f}{\partial z}$

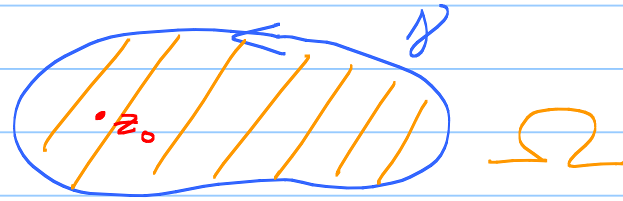
Think: $f = \sum_{n,m} a_{nm} x^n y^m = \sum_{n,m} \alpha_{nm} z^n \bar z^m$ $\quad$ f analytic means no $\bar z$ terms.

Complex chain rule: $w = f(\zeta)$ where $\zeta = g(z)$.

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial \zeta}\frac{\partial \zeta}{\partial z} + \frac{\partial w}{\partial \bar\zeta}\frac{\partial \bar\zeta}{\partial z}$$

$$\frac{\partial w}{\partial \bar z} = \frac{\partial w}{\partial \zeta}\frac{\partial \zeta}{\partial \bar z} + \frac{\partial w}{\partial \bar\zeta}\frac{\partial \bar\zeta}{\partial \bar z}$$

Improved Cauchy-Integral Formula:



$$u(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{u(z)}{z - z_0}\,dz + \frac{1}{2\pi i}\iint_\Omega \frac{\frac{\partial u}{\partial \bar z}(z)}{z - z_0}\,dz \wedge d\bar z$$

Pf: Green's formula (which is Stokes' Thm in $\mathbb{R}^2$)

Remark: $dz \wedge d\bar z = (dx + i\,dy) \wedge (dx - i\,dy)$

$$= \underbrace{dx \wedge dx}_{=0} - i\, dx \wedge dy + i \underbrace{dy \wedge dx}_{-dx\wedge dy} + \underbrace{dy \wedge dy}_{=0}$$

$$= -2i\, dx \wedge dy \leftarrow \text{area measure in } \iint.$$

Lars Hörmander's Complex Analysis in several variables, Chapt. 1.

Theorem: Given a complex valued fcn v in $C^\infty(\Omega)$ where Ω is a domain in $\mathbb{C}$, there is a function $u \in C^\infty(\Omega)$ such that $\frac{\partial u}{\partial \bar z} = v$.

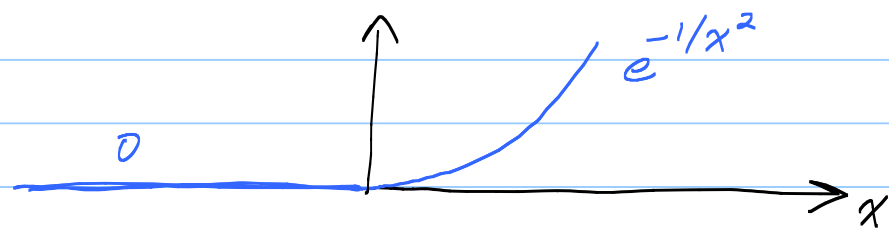
[Can solve the inhomogeneous C-R equns in C^∞.]

Note: If u_1 and u_2 solve the problem, then $\frac{\partial}{\partial \bar z}(\underbrace{u_1 - u_2}) = v - v \equiv 0$

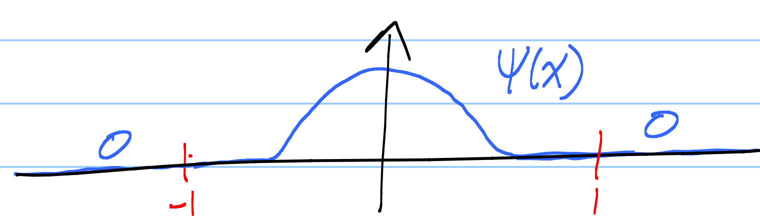
analytic!

Cor: (M-L Thm). $\Omega \subset \mathbb{C}$ is a domain, $\{a_n\}_{n=1}^\infty$ is a seq of distinct discrete pts in Ω and $R_n(z)$ are principal parts at a_n's. Then there is a meromorphic fcn f on Ω with poles only at a_n's with prescribed principal parts (meaning that $f - R_n(z)$ has a removable sing at a_n for each n).

Pf: Cook up C^∞ smooth bump and "cut-off" fcns:



C^∞ smooth $h(x)$



$h(x+\frac{1}{2})\, h(-x+\frac{1}{2})$

Let $c = \int_{-1}^{1} \psi(x)\, dx$

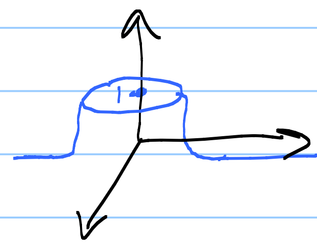
C^∞ bump fcn : $\quad \varphi(x) = \frac{1}{c}\,\psi(x)$

$$\varphi \in C^\infty(\mathbb{R}), \quad \operatorname{supp} \varphi \subset (-1,1), \quad \varphi \geq 0, \quad \int_{-\infty}^{\infty} \varphi \, dx = 1,$$

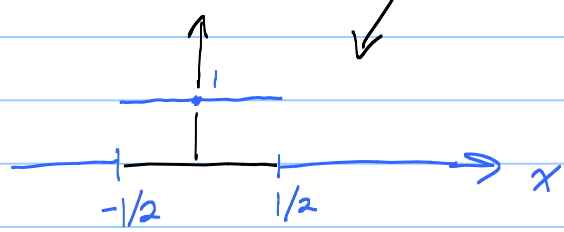
$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\,\varphi\!\left(\frac{x}{\varepsilon}\right) \longleftarrow \text{approximation to the identity}$$

$$\varphi_\varepsilon \in C^\infty(\mathbb{R}), \quad \operatorname{supp} \varphi \subset (-\varepsilon,\varepsilon), \quad \varphi_\varepsilon \geq 0, \quad \int_{-\infty}^{\infty} \varphi_\varepsilon \, dx = 1$$

C^∞ Birthday cake fcn



χ :



$$\psi = \varphi_\varepsilon * \chi = \int_{-\infty}^{\infty} \varphi_\varepsilon(x-y)\,\chi(y)\,dy \quad \text{is} \quad C^\infty\text{-smooth}$$

$$\chi_r^{z_0}(z) = \psi\!\left(\frac{|z-z_0|^2}{r^2}\right) \longleftarrow$$

C^∞ smooth supported in $D_r(z_0)$ and $\equiv 1$ on a nbhd of z_0.

Pf of Cor : Since $\{a_n\}$ discrete, $\exists$ discs $\overline{D_{r_n}(a_n)} \subset \Omega$ such that $\overline{D_{r_n}(a_n)} \cap \overline{D_{r_m}(a_m)} = \emptyset$ if $n \neq m$.

Let $\quad U = \sum_{n=1}^{\infty} \chi_n(z) R_n(z) \quad$ where $\chi_n = \chi_{r_n}^{a_n}$.

Note: U is a C^∞ (non-analytic) sol^n to the M-L problem, and $U \equiv R_n$ near a_n. And U is C^∞-smooth on $\Omega - \{a_n\}_{n=1}^{\infty}$. $\quad [\, U \equiv 0$ outside $\bigcup_{n=1}^{\infty} D_{r_n}(a_n).]$

Let

$$V = \begin{cases} \frac{\partial U}{\partial \bar z} & \text{on } \Omega - \{a_n\}_{n=1}^{\infty} \\ 0 & \text{at } a_n\text{'s} \end{cases}$$

Since $R_n(z)$ is analytic on $D_{r_n}(a_n) - \{a_n\}$, $\frac{\partial U}{\partial \bar z} \equiv 0$ there.

Aha! v is C^∞ smooth on Ω!

Thm $\Rightarrow$ $\exists$ $u \in C^\infty(\Omega)$ with $\frac{\partial u}{\partial \bar z} = v$.

Now $f = U - u$ solves the M-L Problem!

Why: $\frac{\partial f}{\partial \bar z} = \frac{\partial U}{\partial \bar z} - \frac{\partial u}{\partial \bar z} = v - v$ away from a_n's.

So f analytic on $\Omega - \{a_n\}_{n=1}^{\infty}$.

Near a_n: $f = U - u = R_n \chi_n - u = R_n - u$

$f - R_n = u$ near a_n. u $C^\infty \Rightarrow$ bounded near a_n.

So a_n is removable for $f - R_n$.

Facts: M-L and Weierstraß thms hold on any domain in $\mathbb{C}$.

$\mathbb{C}^n$: Solving $\bar\partial$-prob in C^∞ $\Leftrightarrow$ Ω pseudoconvex $\Leftrightarrow$ Ω is a domain of holomorphy.

Fact: $v \in C_0^\infty(\Omega)$

$$u(z) = \frac{1}{2\pi i} \iint_\Omega \frac{v(w)}{w - z}\, dw \wedge d\bar w$$

satisfies $\frac{\partial u}{\partial \bar z} = v$.