Lecture 43 Review

Final Exam, Monday, May 1, 7:00–9:00 pm in HAMP 1144 (Closed books, notes, no electronics as usual.)

5. Let $a_0 = 0$ and $a_1 = 1$. The Fibonacci numbers are defined inductively via

$$a_{n+2} = a_{n+1} + a_n \qquad \text{for } n = 0, 1, 2, \ldots$$

Find the radius of convergence of the power series $\sum a_n z^n$. Hint: Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and compare the power series for $z^2 f(z)$, $z f(z)$, and $f(z)$. Find a closed form formula for f. When you get the picture, make sure everything you say is true. (For example, don't say that $f(z)$ is defined someplace until you know it is.)

Assume f converges. $f(z) - z f(z) - z^2 f(z) = z$ $(*)$

See f has to be $f(z) = \dfrac{z}{1 - z - z^2}$ ← has a conv power series with R. of C. = |nearest zero of denom|

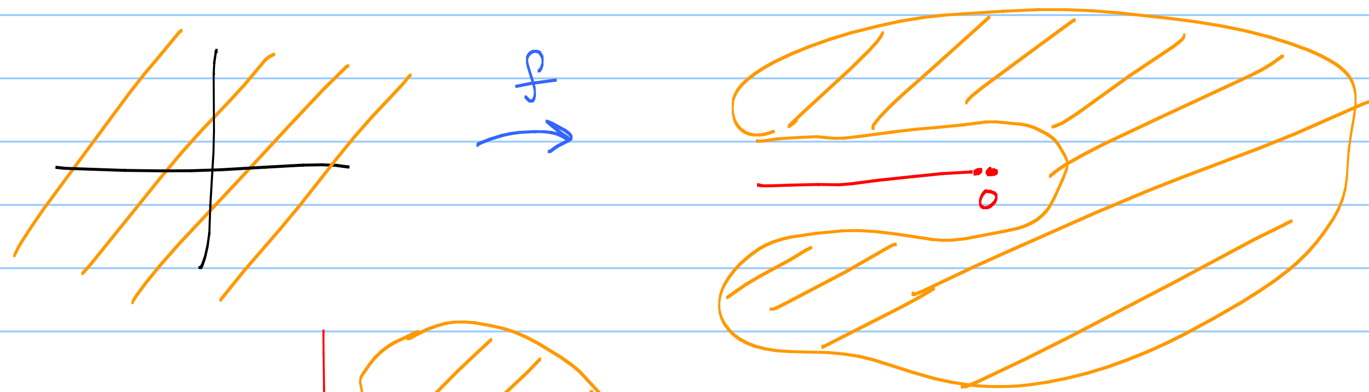
Get $f(z) = \sum_{n=0}^{\infty} b_n z^n$, R. of C. = (Golden ratio related number)

$f(0) = 0 = b_0$

$f'(0) = 1 = b_1$

$(*)$ shows that $b_{n+2} = b_{n+1} + b_n$. So $b_n = a_n \ \forall n$.

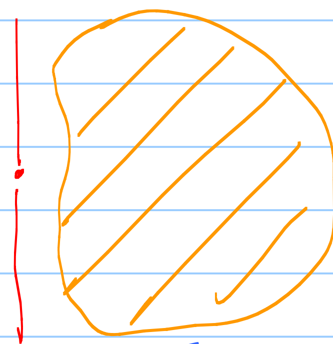
4. Show that if f is an entire function that never takes values along the negative real axis, then f must be a constant function. (You may use any result that was covered in class *except* the Picard Theorems.)



$\xrightarrow{\sqrt{z}}$

$e^{\frac{1}{2}\mathrm{Log}\, z}$

principal $\sqrt{\ }$

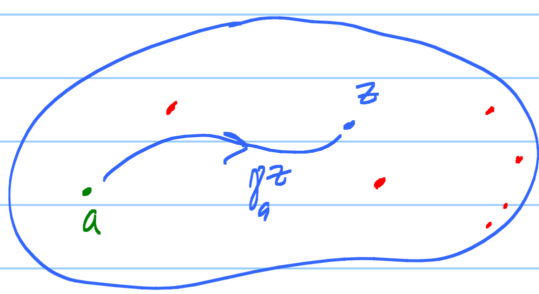


Casorati-Weierstraß $\Longrightarrow$ $\sqrt{f} \equiv c$.

So $f \equiv c^2$

Or $e^{-\sqrt{f}}$, or $\dfrac{\sqrt{f(z)}-1}{\sqrt{f(z)}+1}$ are bnd entire.

8. Suppose that f is a meromorphic function on a simply connected domain Ω that satisfies the following properties. The zeroes of f, if any, have even multiplicity. The poles of f, if any, have even order. Prove that f has a meromorphic square root on Ω.



Try $F(z) = e^{\frac{1}{2}\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\,dw}$

where γ_a^z avoids poles and zeroes of f.

$$\frac{F(z)}{\tilde{F}(z)} = e^{\frac{1}{2}\int_\Gamma \frac{f'}{f}\,dw} = e^{\frac{1}{2}\left(2\pi i \sum\limits_{\substack{\text{zeroes}\\ \text{and}\\ \text{poles}\\ \text{of } f}} \underbrace{\mathrm{Ind}_\Gamma(a)}_{\in \mathbb{Z}} \underbrace{\mathrm{Res}_a \frac{f'}{f}}_{= \begin{cases} 2m & \text{zero} \\ -2m & \text{pole}\end{cases}}\right)}$$

$$= e^{2\pi i \cdot N} = 1$$

See $F'(z) = \frac{1}{2}\frac{f'(z)}{f(z)}F(z)$. $\frac{F^2}{f} = c$, Correct by a const.

6. *(30 pts.)* Suppose that $a_1 = -1$, $a_2 = 1$, and $a_3 = 2i$ and that f is a function that is analytic on $\mathbb{C} - \{a_1, a_2, a_3\}$ that has essential singularities at the three points. Suppose also that

$$\int_{C_1(a_n)} f\,dz = \sqrt{n} \quad \text{for } n = 1, 2, 3,$$

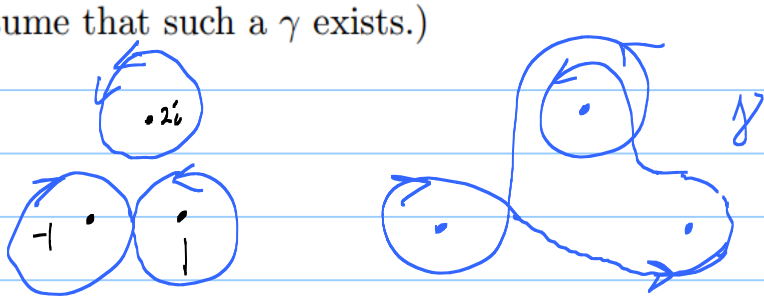
where $C_1(z_0)$ denotes the circle of radius 1 about z_0 parametrized in the counter clockwise sense. Draw a closed curve γ such that

$$\mathrm{Ind}_\gamma a_1 = -1, \qquad \mathrm{Ind}_\gamma a_2 = 1, \quad \text{and} \quad \mathrm{Ind}_\gamma a_3 = 2.$$

Explain how to define a cycle Γ so that the General Cauchy Theorem on the domain $\Omega = \mathbb{C} - \{a_1, a_2, a_3\}$ can be used to compute

$$\int_\gamma f\,dz.$$

Find the value of the integral and explain your reasoning. (You are not allowed to use the General Residue Theorem here. If you failed to draw such a γ, you may assume that such a γ exists.)



Idea: $\Gamma = \gamma \cup \left(-n_1 C_1(a_1) \cup -n_2 C_1(a_2) \cup -n_3 C_1(a_3)\right)$

$n_j = \mathrm{Ind}_\gamma(a_j)$

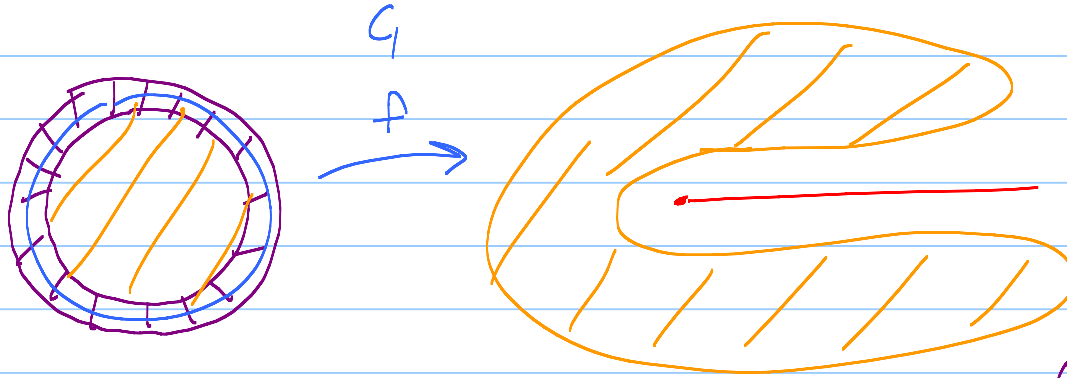
General Cauchy: $\Omega = \mathbb{C} - \{a_1, a_2, a_3\}$.

$0 = \int_\Gamma f\,dz = \int_\gamma - \left(n_1 \mathrm{Res}_{a_1} f + n_2 \mathrm{Res}_{a_2} f + n_3 \mathrm{Res}_{a_3} f\right) \cdot 2\pi i$

Get $\int_\gamma f\,dz = 2\pi i\left[-1\cdot\sqrt{1} + 1\cdot\sqrt{2} + 2\cdot\sqrt{3}\right]$

2. *(30 pts.)* Suppose that f is analytic in a neighborhood of the closed unit disc and that $f(z)$ is never in the set $\{x \in \mathbb{R} : x \geq 0\}$ when $|z| = 1$. Show that f has no zeroes in the unit disc.

$$\#\text{ zeroes of } f = \frac{1}{2\pi i}\int_{C_1} \frac{f'}{f}\,dz = \frac{1}{2\pi i}\int_{C_1} \frac{d}{dz}\left(\log f\right) dz = 0$$



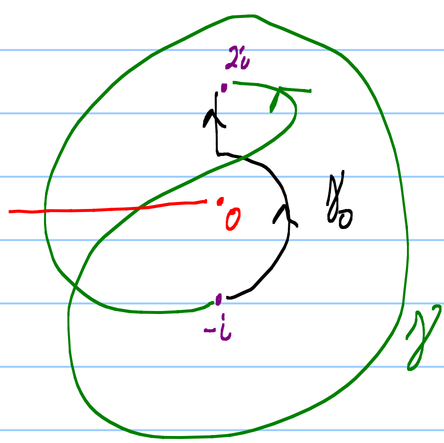
Only need condition on nbhd $C_1(0)$

$\log z = \operatorname{Ln}|z| + i\theta$ where $0 < \theta < 2\pi$, $\theta \in \arg z$

3. What are the possible values of

$$\int_\gamma \frac{1}{z}\,dz,$$

where γ is a path that starts at $z = -i$ and ends at $z = 2i$ and that avoids the origin? Explain.



$$\int_{\gamma_0} \frac{1}{z}\,dz = \int_{\gamma_0} \frac{d}{dz}\left(\operatorname{Log} z\right) dz = \operatorname{Log} 2i - \operatorname{Log}(-i)$$

$$= \left(\operatorname{Ln}2 + i\frac{\pi}{2}\right) - \left(\operatorname{Ln}1 - i\frac{\pi}{2}\right) = \operatorname{Ln}2 + i\pi$$

$\Gamma = \gamma_0 \cup (-\gamma)$ is a closed curve.

$$\left(\int_{\gamma_0} - \int_\gamma\right) \frac{1}{z}\,dz = 2\pi i\, \underbrace{\operatorname{Ind}_\Gamma(0)}_{n \in \mathbb{Z}}\, \underbrace{\operatorname{Res}_0 \frac{1}{z}}_{1}$$

Last step. Cook up γ to show you can get any $n \in \mathbb{Z}$.

<u>Prob</u>: Suppose $F(z,w)$ is <u>continuous</u> on $D_1(0) \times \Omega$ and for fixed w, $F(z,w)$ is analytic in z.

Show that $\displaystyle\int_\gamma F(z,w)\,dw = h(z)$ is analytic on $D_1(0)$ when γ is a curve in Ω.

Morera's! + Fubini. $\Delta \subset D_1(0)$

$$\int_\Delta \underbrace{\left(\int_\gamma F(z,w)\,dw\right)}_{\text{cont. via unif cont on compacts}} dz \overset{\text{Fub.}}{=} \int_\gamma \underbrace{\left(\int_\Delta F(z,w)\,dz\right)}_{\substack{=0 \\ \text{Goursat}}} dw = 0$$

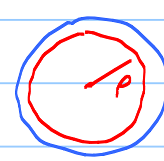
<u>Prob</u>: Suppose $f: D_1(0) \to D_1(0)$, f analytic, <u>and</u>

(*) $\displaystyle\lim_{|z|\to 1} |f(z)| = 1$. [$f^{-1}(K)$ is compact when K is compact. "Proper"]

Then f is a "finite Blaschke product"

$$\varphi(z) = \lambda \prod_{n=1}^{N} \left(\frac{z-a_n}{1-\bar{a}_n z}\right)^{m_n}, \quad a_n \in D_1(0), \quad |\lambda| = 1.$$

(*) + Identity Thm show f has finitely many zeroes a_n, mult m_n. Use Max Princ on $\frac{f}{\varphi}$



$\rho \nearrow 1$

Get $\left|\frac{f}{\varphi}\right| \le 1$. Repeat for $\frac{\varphi}{f}$. Get $\frac{f}{\varphi} \equiv \lambda$.