Lecture 5

HWK 2 on 530 Home page, due week from Monday

Suppose $f(z) = \sum_0^\infty a_n (z - z_0)^n$ on $D_R(z_0)$ with R. of C. = R.

$\lim_{n \to \infty} \sqrt[n]{n} = 1$. So Hadamard's $\Rightarrow$ R. of C. of

$$\sum_{n=1}^\infty n a_n (z - z_0)^{n-1} \text{ has same R. of C.} = R.$$

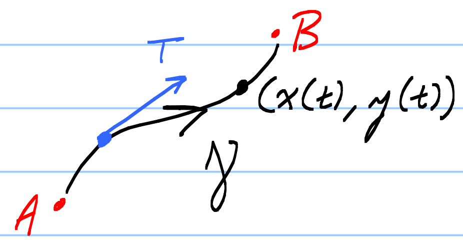
Big fact: f is complex diff'ble on $D_R(z_0)$ and

$$f'(z) = \sum_1^\infty n a_n (z - z_0)^{n-1} \text{ also with R of C} = R. \qquad \text{[Proof later]}$$

Stein: Bare hands proof pp. 16–18.

Bigger fact: If f is analytic on $D_r(z_0)$, then f = conv power series on $D_r(z_0)$ with R of C $\geq r$. [Proof later]

Integration along paths:



$\sim z(t) = x(t) + i y(t)$

$a \leq t \leq b$

$z : [a, b] \to \mathbb{C}$

Def^n: $z'(t) = x'(t) + i y'(t)$

Easy fact: $z'(t_0) = \lim_{t \to t_0} \dfrac{z(t) - z(t_0)}{t - t_0}$

Tangent vector $T \sim z'(t)$ Unit complex tang vect $= \dfrac{z'(t)}{|z'(t)|}$

Def^n: Trace of γ = $tr(\gamma) = \{z(t) : a \leq t \leq b\}$

Chain Rule #1: $g : \Omega_1 \to \Omega_2$ (open sets in $\mathbb{C}$)

$f : \Omega_2 \to \mathbb{C}$

If g analytic on Ω_1 and f on Ω_2, then $h(z) = f(g(z))$ is analytic on Ω_1 and $h'(z) = f'(g(z))\, g'(z)$.

Pf: $\dfrac{f(w) - f(A)}{w - A} = f'(A) + E(w)$ where $E(w) \to 0$ as $w \to A$.

Define $E(A) = 0$.

$$f(w) - f(A) = f'(A)(w-A) + E(w)(w-A)$$

$$\underset{w=g(z)}{\uparrow} \qquad \underset{A=g(a)}{\uparrow}$$

$$\frac{f(g(z)) - f(g(a))}{z-a} = f'(g(a))\,\frac{(g(z)-g(a))}{z-a} + E(g(z))\,\frac{(g(z)-g(a))}{z-a}$$

Let $z \to a$:

$$f'(g(a)) \cdot g'(a) + \underbrace{E(g(a))}_{0} \cdot g'(a)$$

Note: g diff'ble at $a \Rightarrow g(z) \to g(a)$ as $z \to a$.

So $E(g(z)) \to E(g(a)) = 0$.

Chain Rule #2: f analytic and $z(t) = x(t) + i\,y(t)$ is diff'ble.

Then $$\frac{d}{dt} f(z(t)) = \underbrace{f'(z(t))}_{\text{complex dev.}} \cdot z'(t)$$

Pf: Repeat Chain Rule #1 proof, or

$$f(z(t)) = u(x(t), y(t)) + i\,v(x(t), y(t))$$

Use $\mathbb{R} \to \mathbb{R}^2$ chain rule

Use C-R eqns!

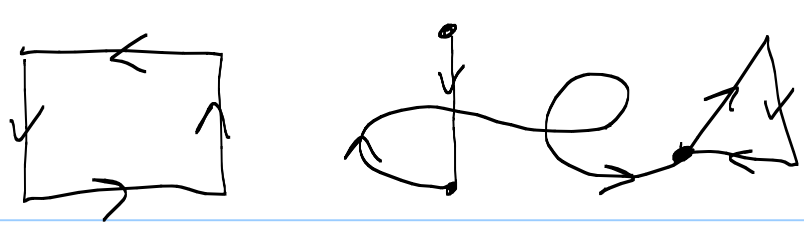
$$\frac{d}{dt} f(z(t)) = [u_x \dot{x} + u_y \dot{y}] + i\,[v_x \dot{x} + v_y \dot{y}]$$

$$= [u_x + i v_x] \cdot [\dot{x} + i \dot{y}]$$

Piecewise $\mathcal{C}^1$-paths:

$t_0 = a \quad t_1 \quad t_2 \quad \cdots \quad b = t_N$

$z(t)$ cont on $[a,b]$, and $z'(t)$ exists and is cont on (t_i, t_{i+1}) and extends cont to $[t_i, t_{i+1}]$.



(Corners but no jumps)

Def^n: $$\int_a^b z(t)\,dt = \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$$

Fundamental Thm of Calculus #1: If $z(t)$ is piecewise C^1, then $$\int_a^b z'(t)\,dt = z(b) - z(a).$$

Pf: $$\sum_{i=1}^{N} \overset{\shortparallel}{\int_{t_{i-1}}^{t_i}} z'(t)\,dt = \sum\left(\int \text{real} + i\int \text{real}\right) \leftarrow \text{FTC } \mathbb{R}$$

$$= \sum_{i=1}^{N} \left[z(t_i) - z(t_{i-1})\right] \leftarrow \text{tele-scopes}$$

$$= z(b) - z(a) \quad \checkmark$$

Lemma: $$\left|\int_a^b z(t)\,dt\right| \le \int_a^b |z(t)|\,dt$$

Pf: Let $c = \int_a^b z(t)\,dt = re^{i\theta}$. Let $\lambda = e^{-i\theta} = \frac{\bar{c}}{|c|}$

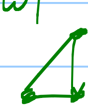
$$\underbrace{\left|\int_a^b z(t)\,dt\right|}_{\text{real !}} = \lambda\int_a^b z(t)\,dt \underset{\substack{\uparrow \\ \text{show this}}}{=} \int_a^b \lambda z(t)\,dt$$

$$= \int_a^b Re[\lambda z(t)]\,dt + i\underbrace{\int_a^b Im[\lambda z(t)]\,dt}_{\text{must} = 0}$$

$$= \int_a^b Re[\lambda z(t)]\,dt$$

$$\le \int_a^b \left|Re[\lambda z(t)]\right|\,dt$$

$|Re\, w| \le |w|$



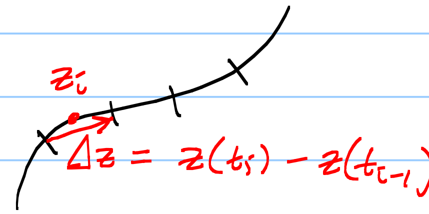
$$\le \int_a^b \underbrace{|\lambda z(t)|}_{=|\lambda|\,|z(t)|} \, dt = \int_a^b |z(t)|\, dt \quad \checkmark$$

$|\lambda| = 1$

<u>Defn</u>: Suppose $f : tr(\gamma) \to \mathbb{C}$ is continuous. Then

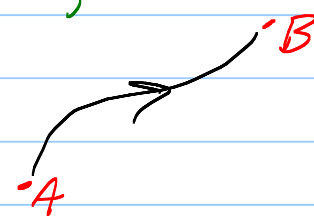
$$\int_\gamma f(z)\, dz = \int_a^b f(\underbrace{z(t)}_{z}) \cdot \underbrace{z'(t)\, dt}_{dz}$$

Konrad Knopf : $\displaystyle = \lim_{|P| \to 0} \sum_{i=1}^{N} f(z_i)\, \Delta z$



<u>Fund Thm Calc #2</u>: If f is analytic on an open set containing $tr(\gamma)$ and f' is continuous there, then

$$\int_\gamma f'\, dz = f(\text{END}) - f(\text{START})$$
$$= f(B) - f(A)$$



<u>Remark</u>: Later, show f analytic $\Rightarrow f'$ anal $\Rightarrow f''$ anal $\Rightarrow \ldots$

(Remove red stuff.) $\Rightarrow f'$ cont

<u>Pf</u>: $\displaystyle \int_\gamma f'\, dz = \int_a^b \underbrace{f'(z(t))\, z'(t)}_{\frac{d}{dt} f(z(t))}\, dt = f(z(b)) - f(z(a))$

Fund Thm Calc #1

<u>Lemma</u>: (Basic Estimate)

$$\left| \int_\gamma f\, dz \right| \le \left(\max_{tr(\gamma)} |f| \right) \cdot \text{Length}(\gamma)$$

where $\displaystyle \text{Length}(\gamma) = \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt \leftarrow$ Defn