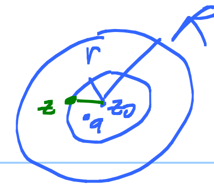
Lecture 8



Thm: f analytic on $D_R(z_0)$, $0<r<R$

$$f'(a) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(z)}{(z-a)^2}\,dz$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_{C_r(z_0)} \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Cauchy Formulas

f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analytic $\Rightarrow \cdots$

Cauchy Estimates: Let $z_0 = a$, $M = \max_{C_r(z_0)} |f|$,

Then $$|f^{(n)}(z_0)| \leq \frac{n!\,M}{r^n}$$

Pf: $$|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right|$$

$$\leq \frac{n!}{2\pi}\left(\max_{C_r} \frac{|f|}{r^{n+1}}\right)\cdot 2\pi r = \frac{n!\,M}{r^n} \checkmark$$

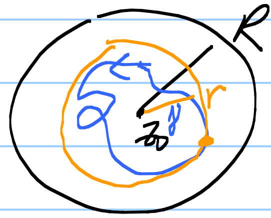
Note: $|f'(z_0)| \leq \frac{M}{r}$, $(n=1)$.

Morera's Thm: Suppose f is continuous on an open set Ω and $\int_\gamma f\,dz = 0$ for all closed γ in Ω (or even less, $\int_Q f\,dz = 0$ for all Δ's $\subset \Omega$). Then f is analytic on Ω.

Pf: Restrict to $D_r(z_0) \subset \Omega$. We have the conclusion of Goursat's Lemma. Cook up $F(z) = \int_{L_a^z} f\,dw$ and show $F' = f$. So F analytic. $\Rightarrow F' = f$ analytic!

Cor: Power series converge to analytic fcns inside circle of conv.

Pf: $\sum_0^\infty a_n (z-z_0)^n$ has R. of C. $R > 0$. Let γ be a closed curve in $D_R(z_0)$. $\gamma: z(t)\quad a \le t \le b$.



$$\text{Let } r = \underbrace{\max_{a \le t \le b}}_{\text{compact}} \underbrace{|z(t) - z_0|}_{\text{cont.}} < R$$

Take ρ with $0 < r < \rho < R$. Hadamard's $\Rightarrow$

$$f_n = \underbrace{\sum_{k=0}^n a_k (z-z_0)^k}_{\text{poly!}} \longrightarrow \text{unif to } f \text{ on } D_\rho(z_0).$$

$\int_\gamma f_n\,dz = 0.$

$$\left| \int_\gamma f\,dz \right| = \left| \int_\gamma f\,dz - \underbrace{\int_\gamma f_n\,dz}_{0} \right|$$

$$= \left| \int_\gamma (f - f_n)\,dz \right| \le \underbrace{\left(\max_\gamma |f - f_n| \right)}_{\to 0 \text{ as } n \to \infty \text{ by unif conv on } D_\rho(z_0).} \cdot \text{Length}(\gamma)$$

Note: Uniform limit of cont. fcns is continuous

Morera's $\Rightarrow f$ analytic!

Def^n: Given a seq f_n of continuous fcns on an open set Ω, we say that "f_n converges uniformly to f on compact subsets of Ω" (write $f_n \twoheadrightarrow f$) if, given any compact subset $K \subset \Omega$, and $\varepsilon > 0$, there is an N such that

$$|f(z) - f_n(z)| < \varepsilon \text{ for } z \in K \text{ if } n > N.$$

Exercize: Power series conv unif on compact subsets of $D_R(z_0)$.

Thm: f_n analytic on Ω and $f_n \rightrightarrows f$, then f is analytic.

[Way false $\mathbb{R} \to \mathbb{R}$. Weierstraß: $\sum_{n=1}^{\infty} \frac{1}{2^n} \sin(2^n t)$
$\rightrightarrows$ nowhere diff fcn!]

Pf: f_n analytic $\Rightarrow$ f_n cont $\Rightarrow$ f cont. Just like power series pf above, show $\int_\gamma f\, dz = 0$, γ closed inside $D_R(z_0) \subset \Omega$. Morera's.

Liouville's Theorem: A bounded entire fcn must be const.

Pf: f entire: f analytic on $\mathbb{C}$

f bounded: $|f(z)| < M$ for all $z \in \mathbb{C}$.

Pick $z_0 \in \mathbb{C}$. Cauchy Estimate $n=1$ yields

$$\underbrace{|f'(z_0)|}_{\text{must be zero}} \le \frac{M}{r} \qquad D_r(z_0) \subset \mathbb{C}$$

Can let $r \to \infty$.

z_0 was arbitrary! So $f' \equiv 0$. $\Rightarrow$ f const.

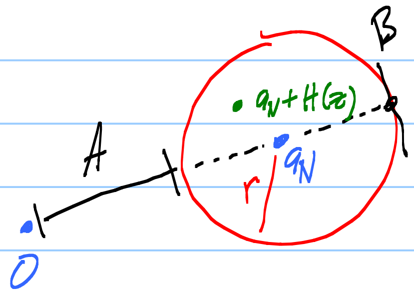
Lemma: Basic polynomial estimate

Suppose $P(z) = a_N z^N + \cdots + a_1 z + a_0$, $N \ge 1$. ($a_N \neq 0$). Then there are constants $0 < A < |a_N| < B$ and a radius R such that $A|z|^N \le |P(z)| \le B|z|^N$ when $|z| > R$.

Remark: A and B can be taken as close to $|a_N|$ as desired.

Pf: $$\frac{P(z)}{z^N} = a_N + \underbrace{\frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N}}_{H(z)}$$

Note: $H(z) \to 0$ as $|z| \to \infty$.



Need $|H(z)| < r$ if $|z| > R$ where could take $r = \frac{|a_N|}{2}$.

Then $A = \frac{|a_N|}{2}$ and $B = \frac{3|a_N|}{2}$.

Thm: Fund Thm Alg. $P(z)$ poly of deg $N \geq 1$.
P has a complex root.

Pf: Suppose not. Then $\frac{1}{P(z)}$ is entire!
Basic Poly estimate shows it is bnd entire.
Liouville's $\Rightarrow \frac{1}{P(z)} = \text{const.}$ ↯ $\left[\frac{d^N}{dz^N} P(z) = N!\, a_N \neq 0 \right]$

Detail: $A|z|^N \leq |P(z)| \leq B|z|^N \quad |z| > R$

$$\left| \frac{1}{P(z)} \right| \leq \frac{1}{A|z|^N} \quad \text{if } |z| > R.$$

So $$\left| \frac{1}{P(z)} \right| \leq \frac{1}{AR^N} \quad \text{if } |z| > R.$$

Also, it is cont on compact $\overline{D_R(0)}$, so it has a max M there.
$\text{Max}\left(M, \frac{1}{AR^N}\right)$ is a bound on $\mathbb{C}$.