Lecture 2 The Cauchy–Riemann equations

Def^n: f (defined on $D_r(a)$) is complex differentiable at a provided that $\lim_{z\to a} \frac{f(z)-f(a)}{z-a}$ exists. Call limit $f'(a)$ if it exists.

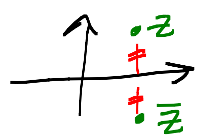
Ex: $f(z)=z^2$ $\quad DQ = \frac{z^2-a^2}{z-a} = \frac{(z-a)(z+a)}{z-a} = z+a \longrightarrow 2a$ as $z\to a$

Write $f'(z)=2z$.

Fact: Basic rules of diff'tiation $\mathbb{R}\to\mathbb{R}$ go over word for word to $\mathbb{C}\to\mathbb{C}$.

Ex: $[fg]' = f'g + fg'$ etc.

Def^n: If $z=x+iy$, then $\bar{z}=x-iy$ is the complex conjugate of z.

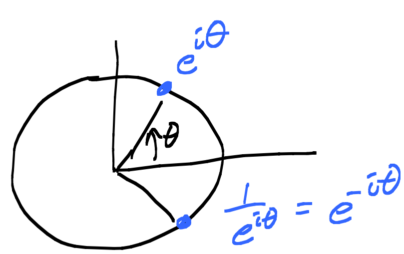


Fact: $\bar{z}$ is not complex diff'ble!

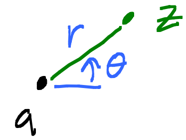
Facts: $|zw|=|z|\cdot|w|$ $\qquad \overline{zw} = \bar{z}\,\bar{w}$

$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ if $w\neq 0$, $\qquad \overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

$e^{i\theta} = \cos\theta + i\sin\theta \qquad \overline{e^{i\theta}} = \cos\theta - i\sin\theta = e^{-i\theta} = \frac{1}{e^{i\theta}}$



Ex: $f(z) = \bar{z}$. $\quad DQ = \dfrac{\bar{z} - \bar{a}}{z - a}$



$r = |z - a|$

$z - a = re^{i\theta}$

$\overline{z - a} = re^{-i\theta}$

$$= \frac{re^{-i\theta}}{re^{i\theta}} = e^{-i2\theta}$$

Ouch! Limit as I slide in along diff radii is different!

Complex limit as $z \to a$ does not exist.

Def^n: f is analytic (or holomorphic) on an open $\Omega \subset \mathbb{C}$ if f is $\mathbb{C}$-diff'ble at every point in Ω.

Def^n: f is continuous at z_0 means $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

Fact: Analytic fcns are continuous.

Pf: $\dfrac{f(z) - f(z_0)}{z - z_0} = f'(z_0) + E(z)$ where $E(z) \to 0$ as $z \to z_0$.

$$f(z) = \underbrace{f(z_0) + f'(z_0)(z - z_0)}_{\mathbb{C}\text{ linear}} + \underbrace{E(z)(z - z_0)}_{\text{very small as } z \to z_0}$$

Let $z \to z_0$: $\quad f(z_0) + f'(z_0) \cdot 0 + 0 \cdot 0 = f(z_0)$ ✓

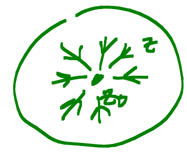
Cauchy-Riemann Eqns: (CR-Eqns) Assume $f(x + iy) = u(x,y) + iv(x,y)$ is $\mathbb{C}$-diff'ble at $z_0 = x_0 + iy_0$. Then, the first partials of u, v exist at (x_0, y_0) and

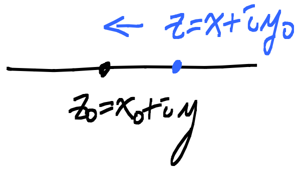
CR-Eqns
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad \text{at } (x_0, y_0)$$

Notation: $u_x = \frac{\partial u}{\partial x}$ $\qquad u^0 = u(x_0, y_0)$

$$u_x^0 = \frac{\partial u}{\partial x}(x_0, y_0)$$

Pf: Step 1: Let $z \to z_0$ along horizontal





$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{[u(x,y_0)+iv(x,y_0)] - u^0 + iv^0}{(x+iy_0)-(x_0+iy_0)}$$

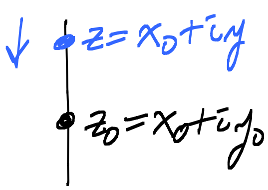
$$= \underbrace{\frac{u(x,y_0)-u(x_0,y_0)}{x-x_0}} + i \underbrace{\frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}}$$

conclude
1st partials exist and

Baby Fact:
$\mathbb{C}$ - limits $\exists$
$\Longleftrightarrow$ Re, Im
limits $\exists$

$$\boxed{f'(z_0) = u_x(x_0,y_0) + i\, v_x(x_0,y_0)}$$ red box #1

Step 2: Vertical next:



$$DQ = \frac{[u(x_0,y)+iv(x_0,y)] - [u^0 + iv^0]}{(x_0+iy)-(x_0+iy_0)}$$

$$= \frac{1}{i} \cdot \frac{u(x_0,y)-u(x_0,y_0)}{y-y_0} + \frac{v(x_0,y)-v(x_0,y_0)}{y-y_0}$$

exist as $y \to y_0$

$$\to -i u_y^0 + v_y^0$$

$$\boxed{f'(z_0) = v_y(x_0,y_0) - i\, u_y(x_0,y_0)}$$ red box #2

box #1 = box #2, C-R Eqns ✓

EX: $E(x+iy) = \underbrace{e^x \cos y}_{u} + i\underbrace{e^x \sin y}_{v}$ $\quad$ u, v satisfy C-R Eqns.

Is $E(z)$ analytic? Yes, because

Partial converse to C-R Eqns: Suppose u and v are C^1-smooth on an open set Ω (meaning u_x, u_y, v_x, v_y exist and u, v and 1st partials are continuous on Ω). If u, v and satisfy C-R Eqns, Then $f(x+iy) = u+iv$ is analytic on Ω.

EX: $\bar{z} = x - iy$ $\quad$ $u(x,y) = x$, $v(x,y) = -y$ $\leftarrow$ C-R Eqns fail.

Looman-Menchoff Thm:

$$\left.\begin{array}{l} \text{1) } u, v \text{ continuous} \\ \text{2) first partials exist} \\ \text{3) C-R Eqns hold} \end{array}\right\} \Rightarrow u+iv \text{ analytic}$$

Narasimhan: Complex analysis in one var. p. 43-50.

Calculus fact: $u(x,y) = u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0) + R(x,y)$

If u is C^1-smooth, then

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

How: $u(x,y) - u(x_0,y_0) = \int_0^1 \frac{\partial}{\partial t}\left[u(x_0 + t(x-x_0), y_0 + t(y-y_0))\right] dt$

$$= \int_0^1 u_x \cdot (x-x_0) + u_y (y-y_0) \, dt$$

$$u - u^0 - u_x^0 (x - x_0) - u_y^0 (y - y_0) = R(x,y)$$

$$= \int_0^1 \underbrace{\left[u_x - u_x^0 \right]}_{\varepsilon} (x - x_0) + \underbrace{\left[u_y - u_y^0 \right]}_{\varepsilon} (y - y_0) \quad dt$$

$$\frac{|\text{eee}|}{|z - z_0|} \le \varepsilon \frac{|x - x_0|}{|z - z_0|} + \varepsilon \frac{|y - y_0|}{|z - z_0|} \le \varepsilon + \varepsilon$$