Lecture 4 Complex power series

Hwk 1 due Friday

Find $E(z) = u + iv$ with $E'(z) = E(z)$:

$$E' = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} \overset{\text{want}}{=} u + iv$$

$$u_x = u : \begin{cases} u = C(y) e^x \\ v = D(y) e^x \end{cases}$$
$$v_x = v :$$

$$\begin{cases} v_y = D'(y) e^x \overset{\text{want}}{=} C(y) e^x \\ -u_y = -C'(y) e^x = D(y) e^x \end{cases}$$

$$\begin{cases} D' = C \leftarrow C' = D'' \\ C' = -D \leftarrow (D'') = -D \leftarrow D'' + D = 0 \end{cases}$$

Get $C'' + C = 0$. $E(0) = 1$ Pins down sol^n!

Lemma: If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

Pf: $S_N = \sum_1^N a_n$. Suppose $S_N \to L$ as $n \to \infty$.

$$a_N = S_N - S_{N-1} \xrightarrow[n\to\infty]{\text{as}} L - L = 0. \checkmark$$

Def^n: $\sum_1^{\infty} a_n$ converges absolutely if $\sum_1^{\infty} |a_n| < \infty$.

Lemma: Abs conv $\Rightarrow$ convergence.

Power series: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$

Big fact: Power series have a radius of conv R

meaning

1) converge inside $D_R(z_0)$

2) converges uniformly in $D_r(z_0)$ when $0<r<R$

3) diverges if $|z-z_0|>R$

$R=0$ case: series only converges at $z=z_0$.

Hadamard's Formula:
$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Def^n:
$$\limsup_{n\to\infty} x_n^{\in \mathbb{R}} = \lim_{N\to\infty} \underbrace{\sup\{x_n : n \geq N\}}_{\substack{\text{non-increasing} \\ \text{as } N \text{ grows}}}$$

So limit as $N\to\infty$ exists.
(Might be $-\infty$.)

Later: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ Hmmm. $\frac{1}{R} = \limsup_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$

Stirling's Formula:
$$\underline{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n!} < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{4n}\right)$$

$$\frac{1}{\sqrt[n]{n!}} < \frac{1}{(2\pi n)^{1/2n}\left(\frac{n}{e}\right)} \longrightarrow 0 \quad \text{as } n\to\infty.$$

Fact: R. of C. of $\sum \frac{z^n}{n!}$ is $R=\infty$.

Geometric series: $\sum_{n=0}^{\infty} z^n$

Ouch: If $|z|\geq 1$, terms don't $\to 0$. So series diverges.

$$\begin{aligned} S_N &= 1+z+\cdots+z^N \\ -\quad zS_N &= \phantom{1+{}} z+\cdots+z^N+z^{N+1} \\ \hline (1-z)S_N &= 1-z^{N+1} \end{aligned}$$

$$S_N = \underbrace{\frac{1}{1-z}}_{\text{limit}} + \underbrace{\left(\frac{-z^{N+1}}{1-z}\right)}_{E_N(z)} \qquad |E_N(z)| = \frac{|z|^{N+1}}{|1-z|}$$

$|z+w| \leq |z| + |w|$ ← numerator estimate

$|z-w| \geq \big||z|-|w|\big|$ ← denominator estimates

$|z+w| \geq \big||z|-|w|\big|$

$|1-z| \geq \big|1-|z|\big| = 1-|z|$ when $|z|<1$.

So $|E_N(z)| \leq \frac{|z|^{N+1}}{1-|z|} \to 0$ as $N\to\infty$

when $|z|<1$

If $|z|<r<1$, then $|E_N(z)| \leq \frac{|z|^{N+1}}{1-|z|} < \frac{r^{N+1}}{1-|z|} < \frac{r^{N+1}}{1-r}$.

Aha! Get uniform conv to $\frac{1}{1-z}$ on $D_r(0)$, $0<r<1$.

Note: $E_N(x) = \frac{-x^{N+1}}{1-x} \to -\infty$ as $x \nearrow 1$.

Do not get uniform conv on $D_1(0)$.

EX: $\sum_{1}^{\infty} \frac{z^n}{n^2}$ converges unif on $\{z : |z| \le 1\}$.

Pf: Pf of Hadamard's. Case $0 < R < \infty$. Assume $z_0 = 0$.

Pick ρ, r with $|z| < r < \rho < R$.

$$\frac{1}{R} \xleftarrow{\text{sliding down to } \frac{1}{R}} \frac{1}{\rho} \qquad \sup_{n \ge N} \sqrt[n]{|a_n|}$$

So $\exists N$ such that $\sqrt[n]{|a_n|} < \frac{1}{\rho}$ if $n \ge N$.

$$|a_n| < \frac{1}{\rho^n} \qquad n \ge N$$

$$|a_n \rho^n| < 1 \qquad n \ge N$$

Hmmm $\overbrace{}^{|z| < r < \rho < R}$

$$|a_n z^n| = \left|a_n \frac{z^n}{\rho^n} \cdot \rho^n\right| = \underbrace{|a_n \rho^n|}_{< 1} \underbrace{\left|\frac{z}{\rho}\right|^n}_{< \left(\frac{r}{\rho}\right)^n} \quad \text{if } n \ge N.$$

Aha! $|a_n z^n| < \left(\frac{r}{\rho}\right)^n$ if $n \ge N$. $\left(\frac{r}{\rho} < 1\right)$

Compare with conv. geom series.

$\sum a_n z^n$ converges if $|z| < r < \rho < R$.

Can make r, ρ, R as close as desired.

Get abs convergence on $D_R(0)$.

Next: Get unif conv on $D_r(0)$, $r < R$.

$$\left| f(z) - \sum^{M} a_n z^n \right| = \left| \sum_{M+1}^{\infty} a_n z^n \right| \le \sum_{M+1}^{\infty} |a_n z^n| \leftarrow \begin{array}{l} |z| < r \\ M > N \end{array}$$

$$\le \sum_{M+1}^{\infty} \left(\frac{r}{\rho}\right)^n = \frac{\left(\frac{r}{\rho}\right)^{M+1}}{1 - \frac{r}{\rho}} \to 0 \text{ as } M \to \infty.$$

Last step: Divergence if $|z| > R$.

$$\frac{1}{|z|} < \frac{1}{R} \le \sup_{n \ge N} \sqrt[n]{|a_n|}$$

sliding down to $\frac{1}{R}$

So, for each N, there is an n with $n \ge N$ such that

$$\frac{1}{|z|} < \sqrt[n]{|a_n|}$$

$$\frac{1}{|z|^n} < |a_n|$$

$$1 < |a_n z^n| \leftarrow$$

Aha! terms $\not\to 0$. Diverges.

EX: $\sum_{n=0}^{\infty} n!\, z^n$ only converges at $z=0$. $R=0$.

Prob: Let $\{a_n\}_{n=1}^{\infty}$ be an enumeration of all the points in $D_1(0)$ with rational real and imag parts.

What is the R. of C. of $\sum_0^{\infty} a_n z^n$?