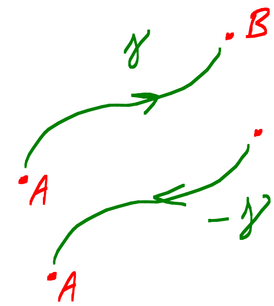
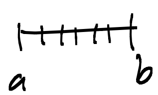
Lecture 6 Goursat's Lemma

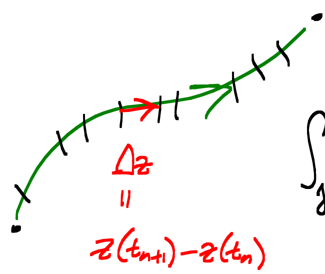
Def^n: $\gamma: z(t) = x(t) + iy(t)$, $a \le t \le b$

$-\gamma: z(b + t(a-b))$: $0 \le t \le 1$



Konrad Knopf:



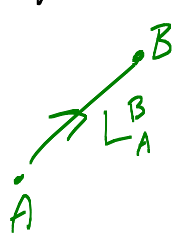


$$\int_\gamma f\,dz \approx \sum f(z(t_i))\,\Delta z \approx -\int_{-\gamma} f\,dz$$

Fact: $\displaystyle\int_{-\gamma} f\,dz = -\int_\gamma f\,dz$

Pf: Write out. Equate real & imag parts. Use one variable change of variables formula.

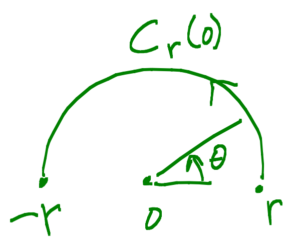
EX:



$z(t) = A + t(B-A)$, $0 \le t \le 1$

$dz = z'(t)\,dt = (B-A)\,dt$

EX:



$z(t) = re^{it}$, $0 \le t \le \pi$

$dz = \underbrace{z'(t)}\,dt = \underbrace{ire^{it}}\,dt$

$$\frac{d}{dt}\left[r\,E(it)\right] = r\,\underbrace{E'(it)}_{e^{it}}\,\underbrace{\frac{d}{dt}[it]}_{i} \quad \checkmark$$

Fact: $\displaystyle\frac{d}{dz}\left[\frac{1}{n+1}z^{n+1}\right] = z^n$

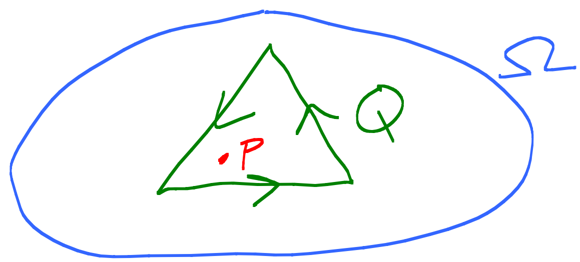
So $\displaystyle\int_\gamma z^n\,dz = \frac{(\text{END})^{n+1}}{n+1} - \frac{(\text{START})^{n+1}}{n+1} = 0$ if γ closed

<u>Consequence:</u> $\int_\gamma \text{Poly}\, dz = 0$, γ closed.

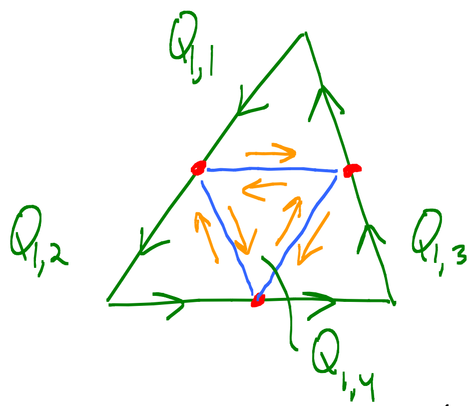
In particular: $\int_\gamma az+b\, dz = 0$, γ closed.

<u>Goursat's Lemma:</u> $\Omega \subset \mathbb{C}$ open, $p \in \Omega$, f continuous on Ω and analytic on $\Omega - \{p\}$, and Q is a $\triangle$ in Ω (and (interior Q) $\subset \Omega$ and $\overline{Q} \subset \Omega$).

Then $\int_Q f\, dz = 0$.



<u>Pf:</u> Case p <u>outside</u> of Q.



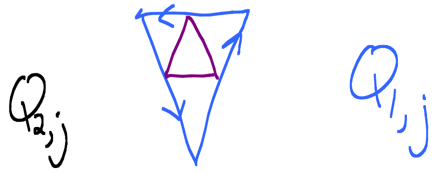
$Q_0 = Q$

Aha! $I = \int_{Q_0} f\, dz = \sum_{j=1}^{4} \int_{Q_{1,j}} f\, dz$

<u>Baby Lemma:</u> If $a_1 + a_2 + a_3 + a_4 = I$, then $\exists\, j$ such that $|a_j| \geq \frac{|I|}{4}$.

Pf: $I = 0$, duh.
$I \neq 0$, Δ ineq. ↯.

$\exists\, j_1$ such that $\left| \int_{Q_{1,j_1}} f \, dz \right| \geq \frac{|I|}{4}$.



Repeat using $Q_{1,j}$.

Get $Q_{2,j}$, $j=1,2,3,4$.

$\exists\, j_2$ such that $\left| \int_{Q_{2,j_2}} f \, dz \right| \geq \frac{|I|}{4^2}$

Keep repeating. Get $Q_0 \supset \underbrace{Q_{1,j_1}}_{Q_1} \supset \underbrace{Q_{2,j_2}}_{Q_2} \supset \underbrace{Q_{3,j_3}}_{Q_3} \supset \cdots$

such that $\left| \int_{Q_n} f \, dz \right| \geq \frac{|I|}{4^n}$

<u>Topology Lemma</u>: Suppose $K_n \neq \emptyset$ are compact $\subset \mathbb{R}^2$ and $K_0 \supset K_1 \supset K_2 \supset K_3 \supset \cdots$. Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.

Furtheromore, if $\mathrm{Diam}(K_n) \to 0$ as $n \to \infty$, then $\bigcap_{n=0}^{\infty} K_n = \{z_0\}$, a single point.

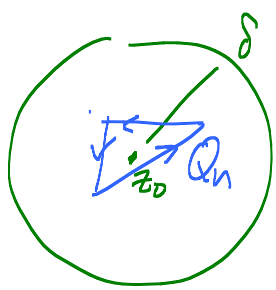
<u>Δ Pf</u>: Take leftmost, or left lower most if more than one, pt of each triangle. x coord non-decreasing. Bdd above. They converge to x_0. Show y-coord also conv. Get z_0.

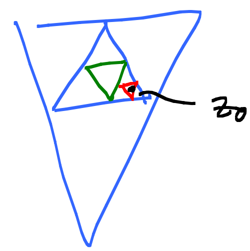
Back to Goursat: $\bigcap_{n=0}^{\infty} Q_n = \{z_0\}$. ← not P

$$\frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) + E(z) \quad \leftarrow \text{Define } E(z_0)=0.$$

$$f(z) = \underbrace{f(z_0) + f'(z_0)(z-z_0)}_{\mathbb{C}\text{-linear}} + \underbrace{E(z)(z-z_0)}_{\text{really small!}} \quad \leftarrow \text{even at } z_0$$

Let $\varepsilon > 0$. There is a δ such that $|E(z)| < \varepsilon$ when $|z-z_0| < \delta$.



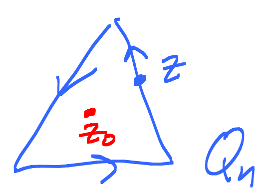


$\exists\, N$ such that $Q_n \subset D_\delta(z_0)$ if $n > N$.

Now $$\int_{Q_n} f\, dz = \underbrace{\int_{Q_n} (\text{Complex linear})\, dz}_{=0} + \int_{Q_n} E(z)(z-z_0)\, dz$$

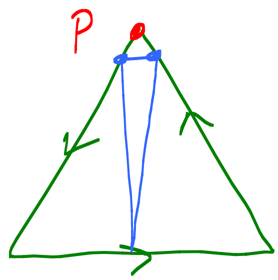
$$\frac{|I|}{4^n} \le \left| \int_{Q_n} f\,dz \right| \le \left| \int_{Q_n} E(z)(z-z_0)\,dz \right|$$

$$n>N \nearrow \quad \le \left(\underbrace{\max_{Q_n} |E(z)|}_{<\varepsilon} \; \underset{\uparrow \atop \frac{\text{Diam}(Q_0)}{2^n}}{\text{Diam}(Q_n)} \right) \underbrace{(\text{Length}(Q_n))}_{\uparrow \atop \frac{\text{Length}(Q_0)}{2^n}}$$



Aha! $|I| \le c\varepsilon$ for any ε! So $I = 0$.

Case $p =$ Vertex:



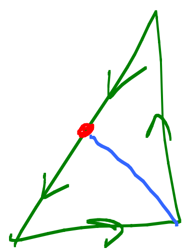
$$\int_\Delta f\,dz = \text{Sum}$$

$$= \int_{\text{Tiny top }\Delta} f \; dz$$

f cont at p, so $|f|$ is Bdd by M on $D_\delta(p)$. Get $\left|\int_\Delta f\,dz\right| \le M\,\text{Length}(\text{Tiny }\Delta)$.

Can make arbitrarily small. So $\int_\Delta f\,dz = 0$.

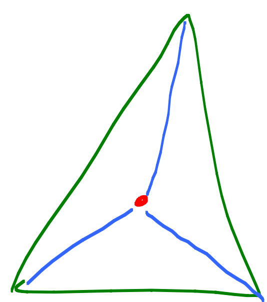
Case $p \in$ side:



reduced to previous case

$$\int_\Delta f\,dz = \int_{\Delta_1} + \int_{\Delta_2} = 0 + 0$$

Case p inside:



reduced to vertex case again.

$$\frac{f(z)-f(p)}{z-p} \longrightarrow f'(p) \quad \text{at} \quad p$$