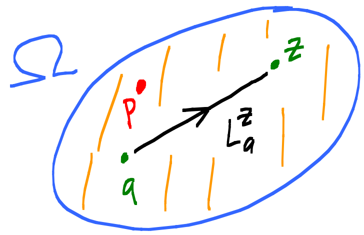
Lecture 7 Cauchy's Theorem on a convex open set

Suppose Ω is a convex open set in $\mathbb{C}$ and suppose f is continuous on Ω and analytic on $\Omega - \{p\}$. Then $\int_\gamma f\, dz = 0$ for any closed path γ in Ω.



Pf: Plan: Given f, cook up F on Ω such $F' = f$ on Ω.
continuous!

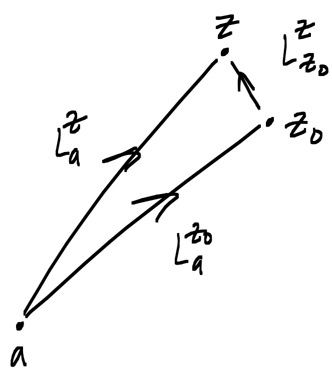
If we had F, then $\int_\gamma f\, dz = \int_\gamma F'\, dz \underset{\text{Fund Thm Calc 2}}{=} F(\text{END}) - F(\text{START})$
$= 0$ if γ closed.

Big idea: If we had F, $\displaystyle\int_{L_a^z} f\, dw = \int_{L_a^z} F'\, dw = F(z) - \underbrace{F(a)}_{\text{const.}}$

Aha! $\displaystyle\frac{d}{dz}\left(\int_{L_a^z} f\, dw\right) = F'(z) = f(z).$

Define $\displaystyle F(z) = \int_{L_a^z} f\, dw.$ Must show $F' = f$.

Key: Goursat



$$\left(\int_{L_a^{z_0}} + \int_{L_{z_0}^z} + \underbrace{\int_{-L_a^z}}_{-\int_{L_a^z}}\right) f\, dw = 0$$

Aha! $DQ = \dfrac{F(z) - F(z_0)}{z - z_0} = \dfrac{1}{z-z_0} \displaystyle\int_{L_{z_0}^{z}} f \, dw$

Little fact: $\displaystyle\int_{L_{z_0}^{z}} c \, dw = \int_{L_{z_0}^{z}} \frac{d}{dw}[cw] \, dw = cw\Big|_{z_0}^{z} = c(z - z_0)$

f continuous at z_0, so $f(z) = f(z_0) + E(z)$ where $E(z) \to 0$ as $z \to z_0$.

$$DQ = \frac{1}{z-z_0} \int_{L_{z_0}^{z}} f(z_0) + E(w) \, dw$$

$$= \underbrace{\frac{1}{z-z_0}\left[f(z_0)(z-z_0) \right]}_{f(z_0)} + \underbrace{\frac{1}{z-z_0} \int_{L_{z_0}^{z}} E(w)\, dw}_{\mathcal{E}(z)}$$

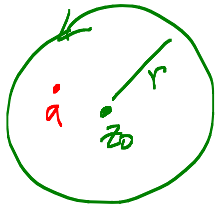
Now $|\mathcal{E}(z)| \le \dfrac{1}{|z-z_0|} \underbrace{\left(\max_{w \in L_{z_0}^{z}} |E(w)| \right)}_{< \varepsilon} \cdot \underbrace{\text{Length}(L_{z_0}^{z})}_{|z - z_0|}$

for $|z - z_0| <$ some δ ✓

Cor: Analytic fcns on a convex open set have an analytic antiderivative there.

Future: Improve "convex open" to $\underbrace{\text{simply connected}}_{\text{no holes}}$ $\underbrace{\text{domain}}_{\text{open connected}}$

Lemma:



$C_r(z_0)$: $z(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$

$$\int_{C_r(z_0)} \frac{1}{z-a}\, dz = \begin{cases} 2\pi i & \text{if } a \in D_r(z_0) \\ 0 & \text{if } |a - z_0| > r \end{cases}$$

by Cauchy's on convex $D_{r+\varepsilon}(z_0)$

Pf: Easy case: $a = z_0$:

$$\int_0^{2\pi} \frac{1}{(z_0 + re^{it}) - z_0} (ire^{it})\, dt = \int_0^{2\pi} i\, dt = 2\pi i \checkmark$$

Sneaky trick: Define $H(w) = \int_{C_r(z_0)} \frac{1}{z-w}\, dz$

Hwk 2: Prob 1: $H'(w) = \int_{C_r(z_0)} \frac{1}{(z-w)^2}\, dz$

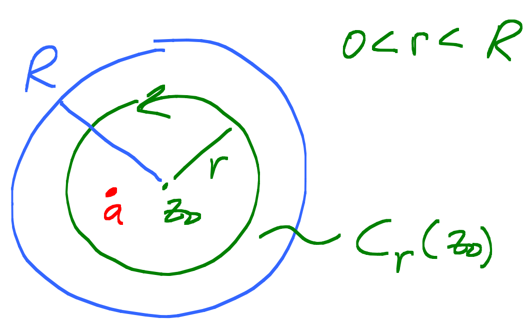
$$\underset{\text{Aha!}}{=} \int_{C_r(z_0)} \frac{d}{dz}\left[\frac{-1}{z-w}\right] dz \underset{\text{Fund Thm Calc \#2}}{=} 0$$

$H' \equiv 0$ on $D_r(z_0) \Rightarrow H \equiv c$, a const. But $H(z_0) = 2\pi i$.

So $H \equiv 2\pi i$ on $D_r(z_0)$. $\checkmark$

Cauchy Integral Formula on a disc

Suppose f analytic on $D_R(z_0)$, $0 < r < R$, and $a \in D_r(z_0)$. Then



$$f(a) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(z)}{z-a}\, dz$$

Pf: $D_R(z_0)$ is a convex open set. $p = a$.

$$G(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

Cauchy on convex:

$$\int_{C_r(z_0)} G\, dz = 0$$

$$0 = \int_{C_r(z_0)} \frac{f(z)-f(a)}{z-a}\, dz$$

$$0 = \int_{C_r(z_0)} \frac{f(z)}{z-a}\, dz - \underbrace{\int_{C_r(z_0)} \frac{f(a)}{z-a}\, dz}_{f(a)\cdot 2\pi i \text{ by Lemma}}$$

Big consequence: Can differentiate under $\int$ in Cauchy Integral Formula as many times as desired. $\Longrightarrow$

f analytic $\Longrightarrow$ f' analytic $\Longrightarrow$ f'' ...

Hwk 2: prob 1:

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-w}\, dz$$

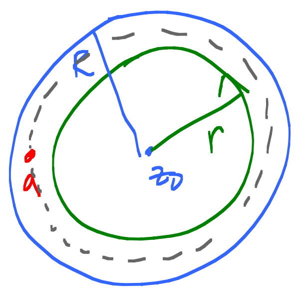
$$f'(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-w)^2}\, dz$$

$$f''(w) = \frac{2}{2\pi i} \int_C \frac{f(z)}{(z-w)^3}\, dz$$

⋮

Higher order Cauchy Formulas $\longrightarrow$ $$f^{(n)}(w) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-w)^{n+1}}\,dz$$

Remark:

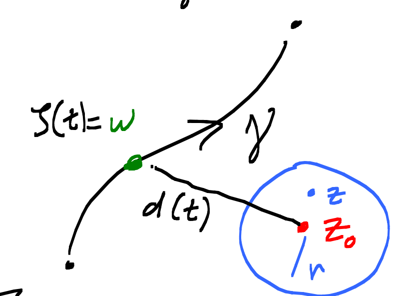


$$\int_{C_r(z_0)} \frac{f(z)}{z-a}\,dz = 0$$ by Cauchy on convex

$\gamma: \zeta(t),\ a \le t \le b$

HWK 2: #1:

$$f(z) = \int_\gamma \frac{\varphi(w)}{w-z}\,dw$$



$r < \frac{\rho}{2}$

$$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{1}{z-z_0}\int_\gamma \varphi(w)\underbrace{\left[\frac{1}{w-z} - \frac{1}{w-z_0}\right]}_{\frac{z-z_0}{(w-z)(w-z_0)}}dw$$

$d(t) = |\zeta(t) - z_0|$

cont fcn + fcn on closed $[a,b]$, so it has a positive min ρ:=

$$= \int_\gamma \frac{\varphi(w)}{(w-z)(w-z_0)}\,dw$$

guess $\longrightarrow$ $$\int \frac{\varphi(w)}{(w-z_0)^2}\,dw$$

prob: Top − Bottom $\longrightarrow 0$ as $z \to z_0$.