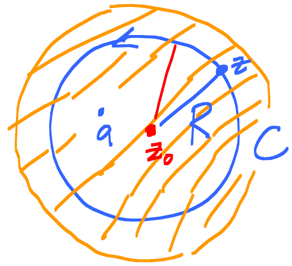
Lecture 8 Liouville's Theorem & the Fundamental Thm Algebra



Cauchy Integral Formulas:

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2}\, dz$$

$\vdots$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Cauchy estimates: Let $a = z_0$. f analytic on an open set containing $D_R(z_0)$.

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi i}\left| \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right|$$

$$\le \frac{n!}{2\pi}\left(\max_{z \in C} \frac{|f(z)|}{R^{n+1}} \right) \underbrace{\text{Length}(C)}_{2\pi R}$$

$$\boxed{|f^{(n)}(z_0)| \le \frac{n!\,M}{R^n}}$$

where $M = \max_{C_R(z_0)} |f|$

$n=1$ case: $|f'(z_0)| \le \dfrac{M}{R}$

Liouville's Theorem: A bounded $\underbrace{\text{entire}}_{\text{analytic on } \mathbb{C}}$ fcn must be constant!

Way false $\mathbb{R} \to \mathbb{R}$. $\sin x$

Pf: "Bounded" means $\exists\ M>0$ such that $|f(z)|<M$ for all z.

Fix $z_0 \in \mathbb{C}$. Cauchy est: $|f'(z_0)| \le \frac{M}{R}$ ← Can let $R \to \infty$.

See $f'(z_0)=0$. Since z_0 was arbitrary, conclude $f' \equiv 0$.

$\Rightarrow f \equiv$ const.

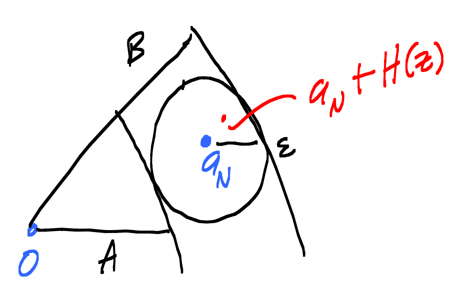
Basic polynomial estimate: Suppose $P(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$ where $N \ge 1$ and $a_N \ne 0$. There exists a radius $R>0$ and real constants A and B with $0 < A < B$ such that

$$A\,|z|^N \le |P(z)| \le B\,|z^N| \quad \text{for } |z|>R.$$

Remark: $A < |a_N| < B$ ← can make them as close as desired.

Pf:
$$\frac{P(z)}{z^N} = a_N + \underbrace{\frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N}}_{H(z)}$$

Notice that $H(z) \to 0$ as $|z| \to \infty$.



Hmmm. Take
$$\varepsilon = \frac{|a_N|}{2}$$

$\exists\ R$ such that $|H(z)| < \varepsilon$ if $|z|>R$.

$A = \frac{|a_N|}{2}$, $B = \frac{3|a_N|}{2}$

$$A \le \frac{|P(z)|}{|z|^N} \le B$$
if $|z|>R$. ✓

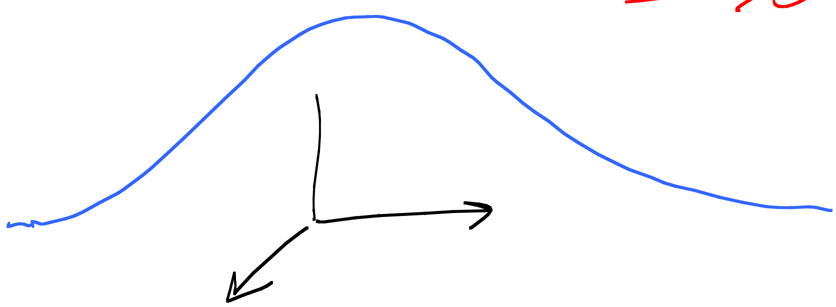
Fundamental Theorem of Algebra: A poly of degree $N \geq 1$ has a root in $\mathbb{C}$. Consequently, can factor polys over $\mathbb{C}$.

Pf: Suppose $P(z)$ does <u>not</u> have a root. Basic est:

$$A|z|^N \leq |P(z)| \leq B|z|^N \quad \text{if} \quad |z| > R.$$

$\frac{1}{P(z)}$ is <u>entire</u>. $\quad \left|\frac{1}{P(z)}\right| \leq \frac{A}{|z|^N}$ if $|z| > R$

$\rightarrow 0$ as $|z| \rightarrow \infty$.



$\exists\, R$ such that $\left|\frac{1}{P(z)}\right| < 1$ if $|z| > R$, and $\left|\frac{1}{P}\right|$ is cont on compact $D_R(0)$. So it has a max M. C: $\max(M, 1)$

$\frac{1}{P(z)}$ is a <u>bnd</u> entire fcn $\Longrightarrow$ $\frac{1}{P(z)} \equiv$ const. $\lightning$. $\quad P^{(N)}(z) = N!\, a_N \neq 0$

Morera's Thm: Suppose f is a continuous complex valued fcn on an open set $\Omega \subset \mathbb{C}$. If $\int_\gamma f\, dz = 0$ for every closed contour γ in Ω, then f is analytic on Ω.

Remark: Enough to know only for $\gamma =$ triangles in Ω.

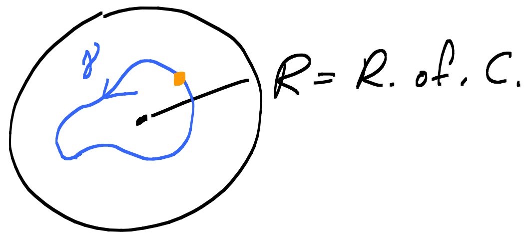
Pf: Restrict attention to a disc $D_r(z_0) \subset \Omega$.

Define $\quad F(z) = \int_{L_a^z} f(w)\, dw$. $\quad$ Aha! We have as hyp the conclusion of Goursat's Lemma. So $F' = f$.

Aha! F is analytic. So $F' = f$ is analytic.

Cor: Power series converge to analytic fcns inside circle of convergence.

Pf:



$tr(\gamma)$ is compact.

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

is a uniform limit of polys on $tr(\gamma)$.

$$\text{But} \quad \int_\gamma P_N(z)\,dz = \sum_0^N a_n \int_\gamma z^n\,dz = 0$$

$$z^n = \frac{d}{dz}\left[\frac{1}{n+1} z^{n+1}\right]$$

$$\left| \int_\gamma f\,dz \right| = \left| \int_\gamma f\,dz - \underbrace{\int_\gamma P_N\,dz}_{0} \right| \le \underbrace{\left(\max_\gamma |f - P_N| \right)}_{\to 0 \text{ as } N \to \infty} \text{Length}(\gamma)$$

So $\int_\gamma f\,dz = 0$ for closed contours γ. Morera's $\Rightarrow$ f analytic

Def^n: Given a seq of fcns f_n of continuous fcns on an open set $\Omega \subset \mathbb{C}$, we say f_n "converges uniformly on compact subsets" of Ω to f if, for each compact $K \subset \Omega$, given $\varepsilon > 0$, there is an N such that $|f(z) - f_n(z)| < \varepsilon$ for all z in K when $n > N$. Write $f_n \twoheadrightarrow f$

Thm: f_n analytic and $f_n \rightrightarrows f$, then f is analytic.

Pf: Restrict to disc. f_n analytic $\Rightarrow$ f_n continuous.

Uniform limits of cont fcns are cont. So f is continuous.

Morera arguement for power series $\Rightarrow$ f analytic.