Lecture 9 Analytic fcns are given locally by convergent power series

Thm: f_n analytic, $f_n \rightrightarrows f$. Then f analytic.

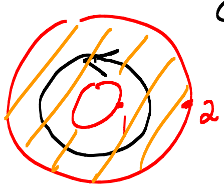
Pf: Restrict to disc (convex ✓). $\int_{\gamma} f_n \, dz = 0$ in disc by Cauchy on convex. ($\gamma \leftarrow$ closed)

$$\left| \int_\gamma f \, dz \right| = \left| \int_\gamma f \, dz - \underbrace{\int_\gamma f_n \, dz}_{=0} \right| \le \underbrace{\left(\max_\gamma |f - f_n| \right)}_{\substack{tr(\gamma) \text{ compact} \\ \to 0}} \text{Length}(\gamma)$$

Morera $\Rightarrow f$ analytic.

Simple fact: On $D_r(a)$, $f_n \rightrightarrows f \iff f_n \to f$ uniformly on each $D_\rho(a)$, $0<\rho<r$.

Danger when a domain has holes: Ex: $f(z) = \frac{1}{z}$ on $\{z : 1 < |z| < 2\}$.



$C_{3/2}(0)$.

$$\int_{C_{3/2}(0)} \frac{1}{z} \, dz = 2\pi i \leftarrow \text{No Cauchy Thm.}$$

And $\frac{1}{z} \neq F'(z)$.

Notation: $S_N(z) = 1 + z + z^2 + \cdots + z^N$

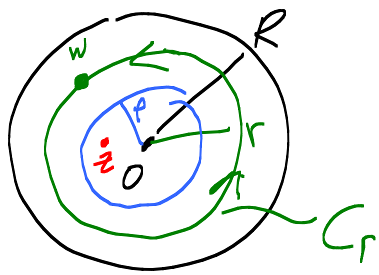
$$\frac{1}{1-z} = S_N(z) + \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$$

If $|z| < \rho < 1$, then $|E_N(z)| < \frac{\rho^{N+1}}{1-\rho}$

Thm: Suppose f analytic on $D_R(z_0)$. Then f is given by a convergent power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ where $a_n = \frac{f^{(n)}(z_0)}{n!}$ and the R. of. C is $\geq R$.

Pf: Can assume $z_0 = 0$ (otherwise let $w = z - z_0$.)

Let $0 < \rho < r < R$.



Note $\left|\frac{z}{w}\right| = \frac{|z|}{r} < \left(\frac{\rho}{r}\right) < 1$

$$f(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w - z}\, dw$$

$$= \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w} \frac{1}{1 - \frac{z}{w}}\, dw$$

$$= \underbrace{\frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w} \left[1 + \left(\frac{z}{w}\right) + \cdots + \left(\frac{z}{w}\right)^N\right] dw}_{\displaystyle \sum_{n=0}^{N} \Bigg(\underbrace{\frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w^{n+1}}\, dw}_{\frac{f^{(n)}(0)}{n!}} \Bigg) z^n} + \underbrace{\frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w} E_N\left(\frac{z}{w}\right) dw}_{\mathcal{E}_N(z)}$$

Yes! if

$$\left|\mathcal{E}_N(z)\right| \le \frac{1}{2\pi} \Big(\underbrace{\max_{C_r} |f|}_{M}\Big) \frac{1}{r} \cdot \frac{\left(\frac{\rho}{r}\right)^{N+1}}{1 - \frac{\rho}{r}} \underbrace{\text{Length}(C_r)}_{2\pi r}$$

$$\le \frac{M \left(\frac{\rho}{r}\right)^{N+1}}{1 - \frac{\rho}{r}} \longrightarrow 0 \quad \text{as } N \to \infty,$$

Series converges unif on $D_\rho(0)$. Can squeeze ρ, r as close to R as desired. So R. of. C is $\geq R$.

Today: $E(z) = e^z = \sum_0^\infty \frac{z^n}{n!}$. R of C $= \infty$.

$$\boxed{\begin{array}{l} E'(z) \\ = E(z), \\ E(0) = 1 \end{array}}$$

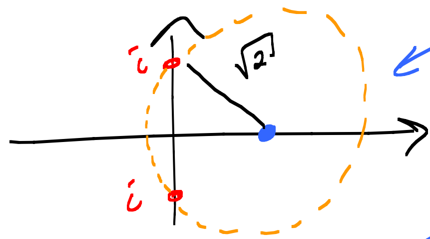
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \qquad R = \infty$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad R = \infty$$

$$\tan z = \frac{\sin z}{\cos z} = \text{power series about } z=0 = \qquad R = ?$$

EX: $f(z) = \frac{1}{z^2+1}$, $z_0 = 1$.

$$= \sum_0^\infty a_n (z-1)^n$$



f analytic on $D_{\sqrt{2}}(1)$.

So $R \geq \sqrt{2}$

But $R \leq \sqrt{2}$ too because $f(z)$ blows up at $\pm i$.

[If power series converged on a bigger disc, see that f does not blow up as approach $\pm i$ from inside disc.]

Thm: Can diff power series term by term: $f = \sum_{n=0}^\infty a_n (z-z_0)^n$ with R. of C. $R > 0$. Then

$$f' = \sum_{n=1}^\infty n a_n (z-z_0)^{n-1} = \sum_{n=0}^\infty \underbrace{(n+1)a_{n+1}}_{b_n} (z-z_0)^n \quad \text{with } \underline{\text{same}} \text{ R. of C.}$$

Pf: $b_n = \dfrac{(f')^{(n)}(z_0)}{n!} = \dfrac{(n+1)\, f^{(n+1)}(z_0)}{(n+1)\cdot n!} = (n+1)\underbrace{\dfrac{f^{(n+1)}(z_0)}{(n+1)!}}_{a_{n+1}}$ ✓

Know f' analytic where f is. And R. of C. $\geq$ radius of biggest disc where analytic.

Easy: $R_{f'} \geq R_f$.

Hmm. If $R_{f'} > R_f$. Cook up analytic antiderivative F for f' on bigger disc. $\frac{d}{dz}[f - F] = 0$.

$f = F + c \leftarrow$ R. of C. too big!

Alternatively: Use Hadamard and $\lim_{n\to\infty} \sqrt[n]{n} = 1$, LimSup facts.

Zeroes of analytic fcns are very polynomialesque.

Suppose f analytic on $D_R(z_0)$ and $f(z_0) = 0 \leftarrow a_0 = 0$

Know $f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n$

Case: All a_n's $= 0$. Then $f \equiv 0$.

Case: Not all zero. Let a_N, $N \geq 1$, be the first non-zero a_n.

$$f(z) = a_N (z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots$$

$$= (z-z_0)^N \Big[\underbrace{\overbrace{a_N + a_{N+1}(z-z_0) + a_{N+2}(z-z_0)^2 + \cdots}^{F(z)}}_{\text{converges for exactly same } z\text{'s as p.s. for } f!} \Big]$$

$F(z)$ is analytic and same R. of C. as p. series for f, and

$$F(z_0) = a_N \neq 0.$$

F analytic $\Rightarrow$ F continuous. So $F(z_0) \neq 0 \Rightarrow$

$\exists\ \delta > 0$ such $F(z)$ is non-vanishing on $D_\delta(z_0)$.

Aha! z_0 is only zero of f inside $D_\delta(z_0)$.

Thm.: Zeroes of analytic fcns must be isolated

(if not $\equiv 0$)...