Lesson 10 Zeroes of analytic functions

Last time: Lemma: Suppose f analytic on $D_R(z_0)$ and $f(z_0)=0$.

Then either A) $f \equiv 0$ on $D_R(z_0)$, or

B) the zero of f at z_0 is isolated (meaning $\exists\ \delta > 0$ with $0 < \delta < R$ such that $f(z) \neq 0$ for $z \in D_\delta(z_0) - \{z_0\}$).

first non-zero a_n

$$f(z) = a_N (z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots$$
$$= (z-z_0)^N \underbrace{\left[a_N + a_{N+1}(z-z_0) + \cdots \right]}_{F(z)}$$

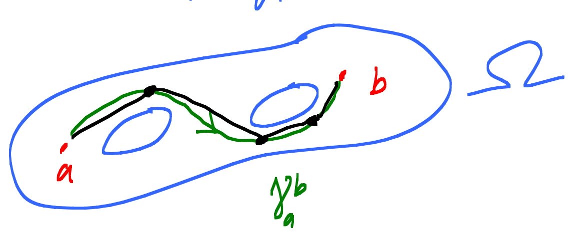
$F(z)$, $F(z_0) = a_N \neq 0$

Def^n: We say that f has a zero of order (or multiplicity) N at z_0,

Connected open sets in $\mathbb{C}$ (Domains)

Def^n #1: Suppose $\Omega \subset \mathbb{C}$ is open. Ω is path-connected if any two points $a, b \in \Omega$ can be joined by a piecewise $\mathcal{C}^1$-smooth curve γ_a^b in Ω. $(tr(\gamma_a^b) \subset \Omega)$

Remark: Topology: γ_a^b is merely taken to be continuous.



Approximate γ cont. by piecewise linear.

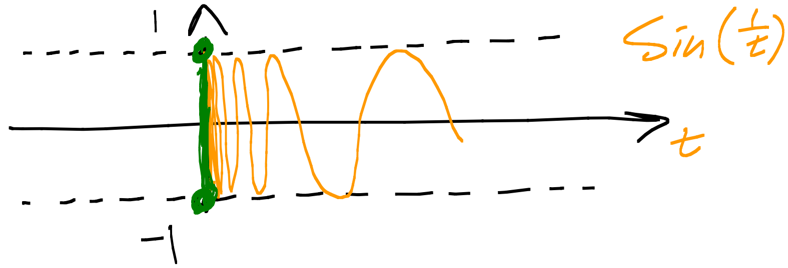
Def^n equiv, $\Omega \subset \mathbb{C}$ open.

Def^n #2: If U_1 and U_2 are open subsets of Ω (open)

and
$$1)\ U_1 \cup U_2 = \Omega$$
$$2)\ U_1 \cap U_2 = \emptyset$$

then either $U_1 = \Omega,\ U_2 = \emptyset$ or $U_1 = \emptyset,\ U_2 = \Omega$.

EX:

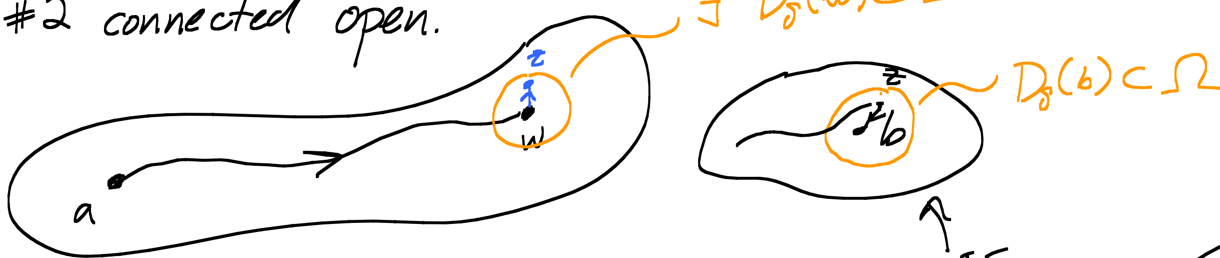


] union graph of $\sin(1/t)$

#2 Connected
not path-connected

Fact: Ω open in $\mathbb{C}$. Def^n #1 $\Longleftrightarrow$ #2.

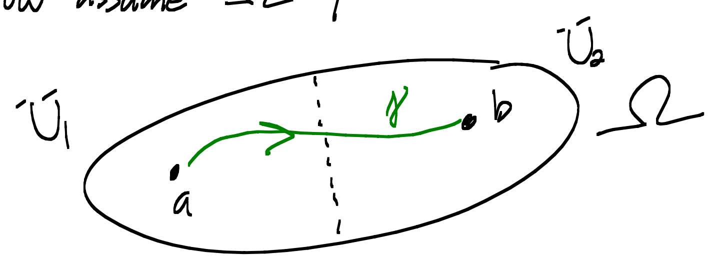
Pf: Ω #2 connected open.



$U_1 = \{ w \in \Omega : \exists\ \gamma_a^w \subset \Omega \}$ open ✓

$a \in U_1$. So $U_2 = \emptyset$. ✓

Now assume Ω path-connected and $\Omega = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$.



γ: $z(t)$, $0 \le t \le 1$

Let
$t_\infty = \sup\{ T : z(t) \in U_1 \text{ for all } t \in [0, T) \}$

Show $z(t_\infty)$ not in either U_1 or U_2. ↯.

Math 530: Analytic fcns on domains

Theorem: f analytic on a domain Ω and $f' \equiv 0$ on Ω, then f is const on Ω.

Pf: $\underbrace{\int_{\gamma_a^b} f'\,dw}_{0} = f(b) - f(a)$ ✓ Fix a. Let b roam.

Theorem (Identity Theorem): Suppose f is analytic on a domain Ω, Let $Z_f = \{z \in \Omega : f(z) = 0\}$. Then, either

A) $f \equiv 0$ on Ω, or

B) Z_f has no limit points in Ω.

Note: z_0 is a limit pt of Z_f in Ω means $\exists$ seq $\{z_n\}_{n=1}^{\infty}$ with $z_n \in Z_f$ and $z_n \neq z_0$ for each n and $z_n \to z_0 \in \Omega$ as $n \to \infty$.

Remark: If $z_0 \in \Omega$ is a limit pt of Z_f, then $f(z_0) = 0$ by continuity. So $z_0 \in Z_f$ too.

Pf of Thm: Suppose Z_f has a limit in Ω.

Let $U_1 = \{z_0 \in \Omega : z_0 \text{ is a limit pt of } Z_f\}$.

Step 1: U_1 open: Suppose $z_0 \in U_1$. $\exists\ D_\varepsilon(z_0) \subset \Omega$. $f(z_0) = 0$

and z_0 is a limit pt of zeroes, so it is <u>not</u> isolated.

Lemma $\Longrightarrow f \equiv 0$ on $D_\varepsilon(z_0)$. So $D_\varepsilon(z_0) \subset U_1$.

<u>Step 2</u>: $U_2 = \Omega - U_1$ is open. Supose $w_0 \in U_2$.

<u>Case $f(w_0) \neq 0$.</u> Then f is non-vanishing on $D_\delta(w_0)$ for some $\delta > 0$, $D_\delta(w_0) \subset \Omega$. See $D_\delta(w_0) \subset U_2$.

<u>Case $f(w_0) = 0$</u>: Ω open. So $\exists\ D_\delta(w_0) \subset \Omega$.

Since w_0 <u>not</u> a limit pt of zeroes. So $f \not\equiv 0$ on $D_\delta(w_0)$. So w_0 is an <u>isolated</u> zero.

$\exists\ \delta'$ such that w_0 is <u>only</u> zero of f on $D_{\delta'}(w_0) - \{w_0\}$,

See $D_{\delta'}(w_0) \subset U_2$.

<u>Done</u>: Assumed Z_f had a limit pt. So $U_1 \neq \emptyset$.

Ω connected $\Longrightarrow U_2 = \emptyset$.

Conclude $U_1 = \Omega$. So $f \equiv 0$ on Ω.

<u>Consequence</u>: e^z is the <u>only</u> analytic extension of e^x from $\mathbb{R}$ to $\mathbb{C}$.

<u>Why</u>: If $f(z)$ and $g(z)$ agree on $\mathbb{R}$, then $f(z) - g(z)$ vanishes on $\mathbb{R} \subset \mathbb{C}$. Every point in $\mathbb{R}$ is a limit pt of $\mathbb{R}$. So $f \equiv g$ on $\mathbb{C}$.

<u>Consequence</u>: Trig identities hold for complex angles.

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \leftarrow$ first let $\alpha = z$, then let $\beta = w \in \mathbb{C}$,

Cor: f, g analytic on a domain Ω and agree on a set with a limit pt in Ω, then $f \equiv g$ on Ω.

Way false $\mathbb{R} \to \mathbb{R}$:

$$h(t) = \begin{cases} e^{-1/t^2}\sin\frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

is a C^∞-smooth fcn. Zeroes pile up at 0.

Also is an example of a C^∞ fcn $\neq$ real power series centered at zero.