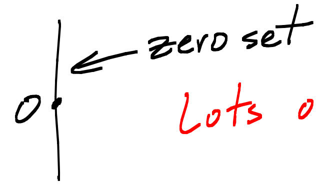
Lecture 12 Complex conjugates & harmonic fcns

HWK 4 on home page due next Friday

Zeroes of harmonic fcns?

EX: $x = \mathrm{Re}\, z \leftarrow$ harmonic



Lots of limit pts!

No identity thm for harmonic, but . . .

Thm: If u is harmonic on a domain Ω and $u = 0$ on $D_\varepsilon(z_0) \subset \Omega$, then $u \equiv 0$ on Ω.

Key: Important trick: Hmmm. Get harmonic conjugate v on $D_\varepsilon(z_0)$.

$f = u + iv$ analytic. Red box #1 : $f' = u_x + i v_x$
Red box #2 : $f' = v_y - i u_y$

$$\left.\begin{array}{l} f' = u_x + i v_x \\ f' = v_y - i u_y \end{array}\right\} = \underbrace{u_x - i u_y}_{u\text{'s only.}}$$

Assume u $\mathcal{C}^2$ smooth and $\Delta u = 0$. Let $\begin{cases} U = u_x \\ V = -u_y \end{cases}$. are $\mathcal{C}^1$-smooth

and U, V satisfy the C-R Eqns. C-R Eqn #1: $\Delta u \equiv 0$
#2: mixed partials equal.

Fact: Given $\mathcal{C}^2$-smooth u with $\Delta u \equiv 0$, get analytic $f = u_x - i u_y$.

Pf of Thm: $u \equiv 0$ on $D_\varepsilon(z_0) \Rightarrow f \equiv 0$ on $D_\varepsilon(z_0) \Rightarrow f \equiv 0$ on Ω,

$\Rightarrow \nabla u \equiv 0$ on Ω. $\Rightarrow u$ const on Ω.

Const must be zero since $u(z_0) = 0$.

Thm: If u is $\mathcal{C}^2$-smooth harmonic on a convex open set, then u has a harmonic conjugate there.

Pf: Let $f = u_x - i u_y$. Let $F = U + iV$ be an analytic antiderivative of f on Ω. $F(z) = \int_{L_{z_0}^{z}} f(w)\,dw$.

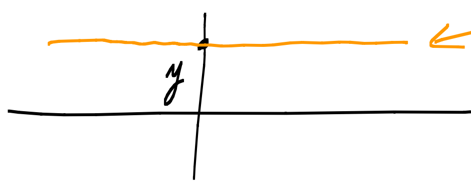
$$F' = U_x - i U_y = f = u_x - i u_y. \quad \text{See } \nabla(U - u) \equiv 0,$$

So $U = u + c$ and we see that V is a harm conj for u,

Lemma: $\frac{\partial u}{\partial x} \equiv 0 \iff u$ fcn of y alone

$\frac{\partial u}{\partial y} \equiv 0 \iff u$ fcn of x alone

Pf:



$\frac{\partial u}{\partial x} \equiv 0$ $u = $ const along horiz lines. ✓

Cor: $\frac{\partial u_1}{\partial x} \equiv \frac{\partial u_2}{\partial x}$, then $u_1 = u_2 + $ (fcn of y alone). etc.

Prob: Find a harmonic conjugate for $u = x^2 - y^2 + x + y$

Harmonic ✓ Look for v with

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad \begin{cases} v_x = -u_y & (A) \\ v_y = u_x & (B) \end{cases}$$

First use (A): Want $v_x = -(0 - 2y + 0 + 1) = 2y - 1$

So $$V = \int 2y - 1 \, dx = 2xy - x + \underbrace{C(y)}_{\text{arb fcn of } y}$$

treat y like a const in $\int$

Cor: $\frac{\partial}{\partial x}\Big(V - (2xy - x)\Big) \equiv 0$. So $C(y)$ belongs there.

Got $$\boxed{V = 2xy - x + C(y)}$$

Next use (B): $V_y \overset{\text{want}}{=} u_x = \frac{\partial}{\partial x}\left(x^2 - y^2 + x + y\right) = 2x + 1$

$$\frac{\partial}{\partial y}\Big(2xy - x + C(y)\Big) = 2x + 1$$

$$2x + C'(y) \overset{\text{want}}{=} 2x + 1$$ ← miracle happens here. All x's cancel.

$$\boxed{C'(y) = 1}$$

Get $C(y) = \int 1 \, dy = y + K$ ← arb const.

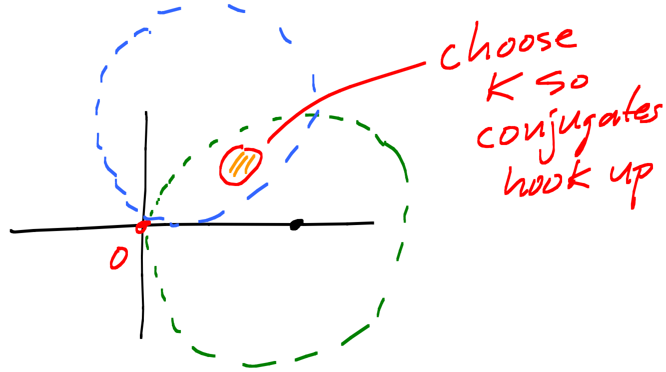
And $$\boxed{V = 2xy - x + (y + K)}$$ ← V uniq up to $+K$.

Fact: u C^2-smooth harmonic. Then u is the real part of an analytic fcn locally [on discs $\subset$ domain]. And so u is C^∞-smooth!

EX: Danger of holes in a domain :

$u = \mathrm{Ln}|z|$ is harmonic.

Can't get a harmonic conj on $\mathbb{C} - \{0\}$.



Complex log fcn: $w = e^z = e^{x+iy} = e^x e^{iy}$

$z \in \log w$

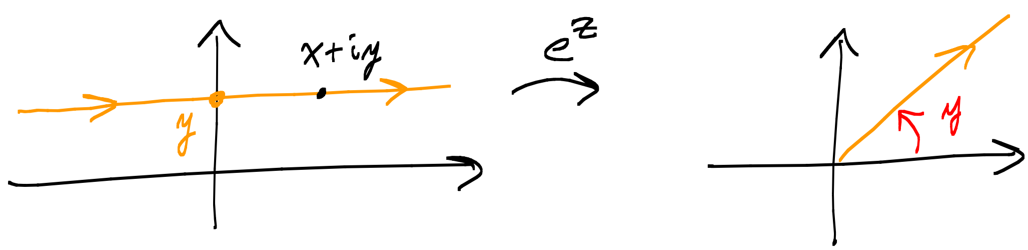
See $e^x = |w|$. So $x = \mathrm{Ln}|w|$ is determined.

But $y \in \arg w$. So $y = \mathrm{Arg}\, w + 2\pi n$, $n \in \mathbb{Z}$

Notation: $\log z = \{ \mathrm{Ln}|z| + i\theta \ : \ \theta \in \arg z \}$ ← set

$\mathrm{Log}\, z = \mathrm{Ln}|z| + i\, \mathrm{Arg}\, z$ ← princ arg z.

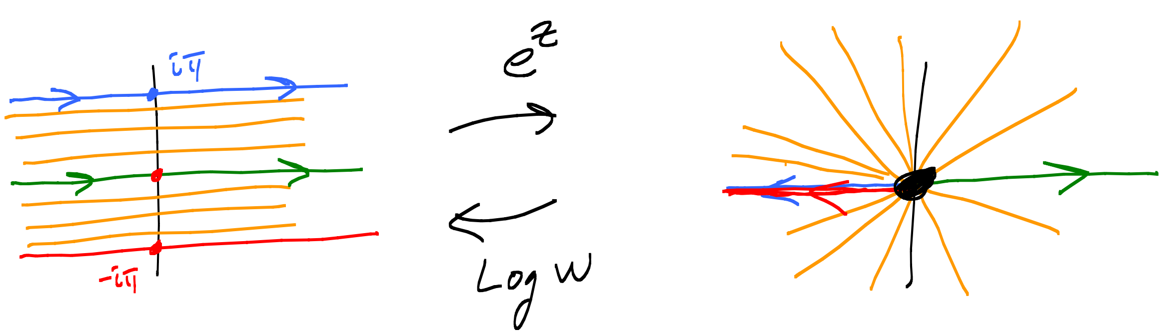
Graph e^z:

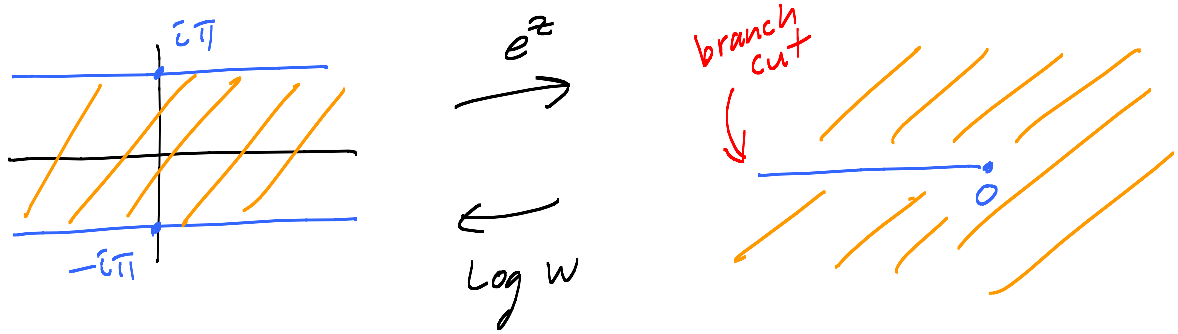


$e^{x+iy} = e^x e^{iy}$

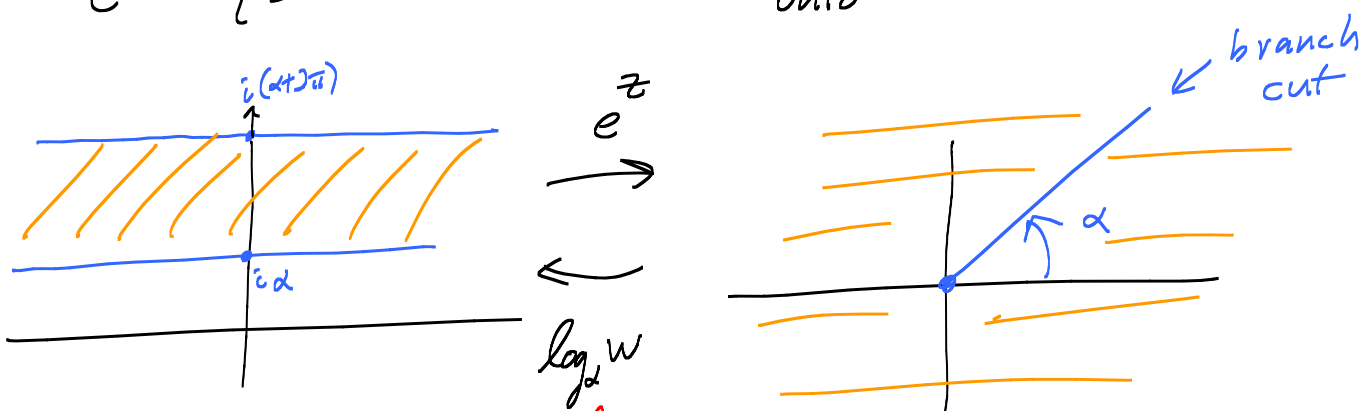
$-\infty < x < \infty$

$0 < e^x < \infty$





$$e^z : \{z : -\pi < \operatorname{Im} z < \pi\} \xrightarrow[\text{onto}]{1-1} \mathbb{C} - (-\infty, 0]$$



branch of complex log fcn.

Think about graph of $e^{1/z}$ near $z=0$.