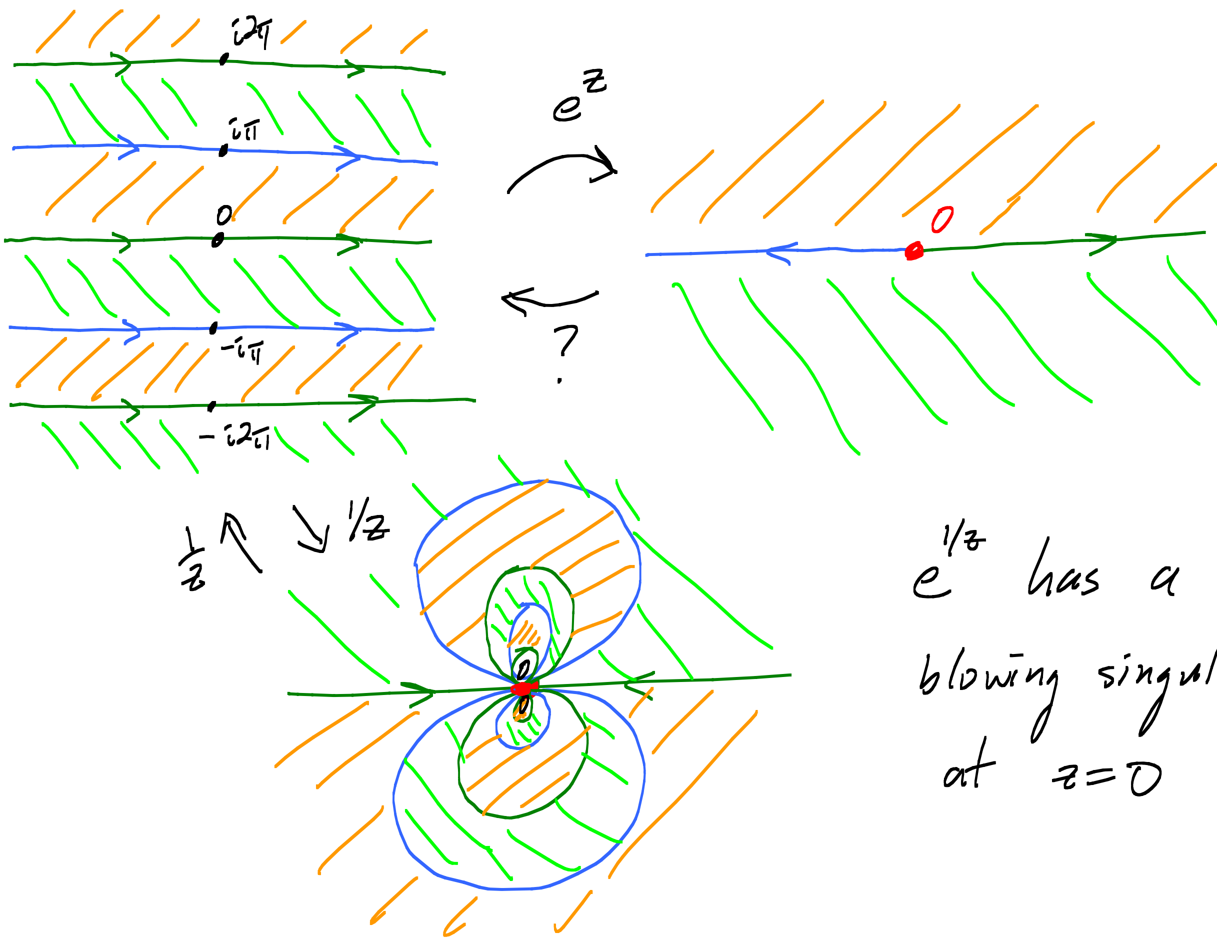
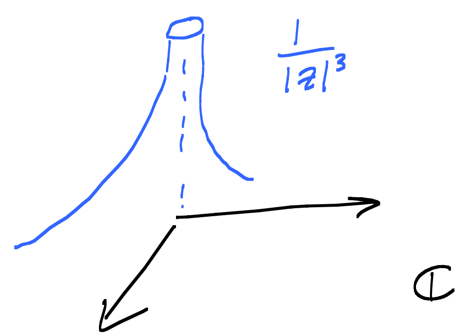
Lecture 13 The Riemann removable singularity theorem



$e^{1/z}$ has a mind blowing singularity at $z=0$

No matter how small $\varepsilon > 0$ is, $e^{1/z}$ maps $D_\varepsilon(0) - \{0\}$ onto $\mathbb{C} - \{0\}$ infinite-to-one! 0 is an "essential singularity" of $e^{1/z}$.

Other singular behavior: 1) $\frac{1}{z^3}$



Note: $\lim_{z \to 0} \left| \frac{1}{z^3} \right| = \infty$

$z = 0$ is a "pole" of $\frac{1}{z^3}$.

2) $$\frac{\sin z}{z} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{z} = \frac{z\left(1 - \frac{z^2}{3!} + \cdots\right)}{z}$$

$= 1 - \frac{z^2}{3!} + \cdots$ ← analytic at $z=0$ (entire!)

$$f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases} \quad \text{is analytic on } D_\varepsilon(0).$$

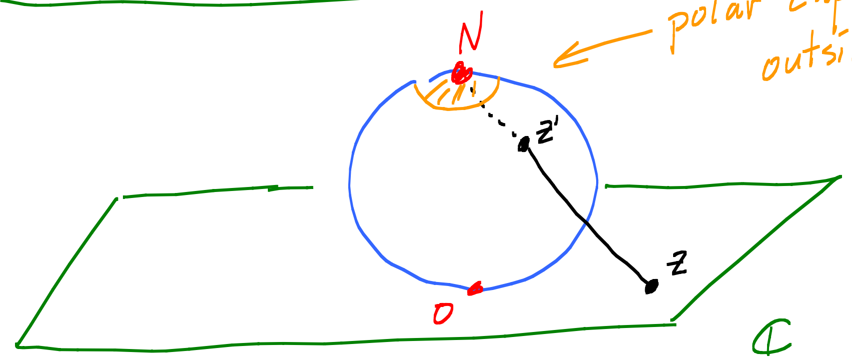
remove sing by defining $f(0)=1$.

$z=0$ is a "removable singularity" of $\frac{\sin z}{z}$.

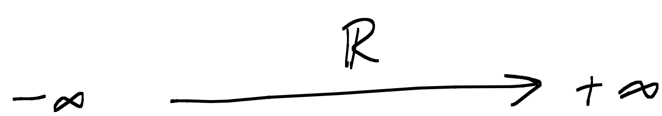
<u>Big fact</u>: These are the only 3 possible behaviors at an "isolated singularity" (meaning $\exists\ \varepsilon > 0$ such that fcn is analytic on $D_\varepsilon(z_0) - \{z_0\}$.)

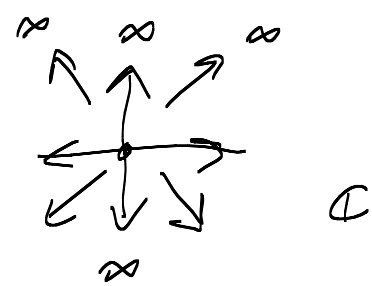
The "point at infinity":

polar cap maps to outside of a big disc



$N \sim \infty$





Extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

A small ε disc about ∞ : $D_\varepsilon(\infty) = \{z : |z| > \frac{1}{\varepsilon}\} \cup \{\infty\}$.

small polar cap if ε small.

Def^n: $\lim_{z \to a} f(z) = \infty$ means, given $M > 0$, there is a $\delta > 0$ such that $|f(z)| > M$ when $|z - a| < \delta$ and $z \neq a$. $\left[\lim_{z \to a} |f| = \infty \right]$

Think: Setting $f(a) = \infty$ makes f "continuous" at a.

Riemann removable singularity thm: Suppose f is analytic on $D_r(a) - \{a\}$ and bounded there, meaning $\exists\, M > 0$ such that $|f(z)| < M$ for $z \in D_r(a) - \{a\}$. Then can give $f(a)$ a value to make f analytic on $D_r(a)$: $z = a$ is removable.

Way false $\mathbb{R} \to \mathbb{R}$: $\sin\left(\frac{1}{t}\right)$

Pf: Define

$$G(z) = \begin{cases} f(z)(z-a) & z \neq a \\ 0 & z = a \end{cases}$$

Note $\lim_{z \to a} G(z) = 0$ because $|f| < M$. $\left[\begin{array}{l} \varepsilon > 0 \\ \delta = \frac{\varepsilon}{M} \end{array} \right]$

Goursat, Morera's $\Longrightarrow$ G analytic on $D_r(a)$.

Aha! $G(z) = \text{power series} = \underbrace{G(a)}_{=0} + G'(a)(z-a) + \cdots$

$$= (z-a)^N H(z) \qquad N \geq 1$$

analytic $H(a) \neq 0$.

Now see $f(z) = \dfrac{G(z)}{z-a} = \underbrace{(z-a)^{N-1} H(z)}$ for $z \neq a$

analytic on $D_r(a)$.
See value $f(a)$ should be. ✓

Alternate pf:

$$G(z) = \begin{cases} (z-a)^2 f(z) & z \neq a \\ 0 & z = a \end{cases}$$

Show $G'(0)$ exists. So G analytic.

Show $G(0) = 0$ ✓ $\quad G'(0) = 0$.

Use power series to factor out $(z-a)^2$. ✓

Def^n: Suppose a is an isolated sing of f analytic on $D_\rho(a) - \{a\}$.

1) a is removable if we can define $f(a)$ to make f analytic on $D_\rho(a)$.

2) a is a pole if $\lim\limits_{z \to a} |f(z)| = \infty$

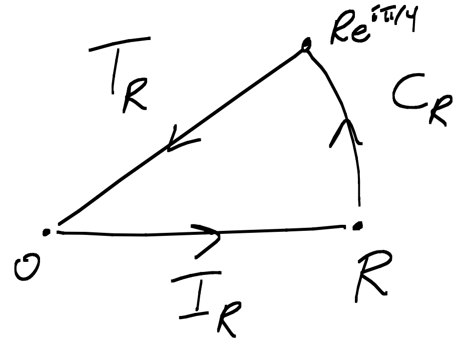
3) a is essential if, for every ε with $0 < \varepsilon < \rho$, $f(D_\varepsilon(a) - \{a\})$ is dense in $\mathbb{C}$, meaning every point in $\mathbb{C}$ is a limit point of

$\{w : w = f(z) \text{ for some } z \in D_\varepsilon(a) - \{a\}\}$

Thm: These are the only 3 possible type of behaviors.

HWK 4 hints:

γ_R



$$\int_{\gamma_R} e^{-z^2}\,dz = 0$$

$R \to \infty$ $\quad I_R : z(t) = t, \quad 0 \le t \le R$

$$\int_{I_R} e^{-z^2}\,dz = \int_0^R e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} \leftarrow \text{known}$$

$$-T_R : \quad z(t) = t e^{i\pi/4}, \quad 0 \le t \le R.$$

$$\int_{T_R} = -\int_{-T_R} = \sim\sim \leftarrow \text{show related to } \int \text{ you want.}$$

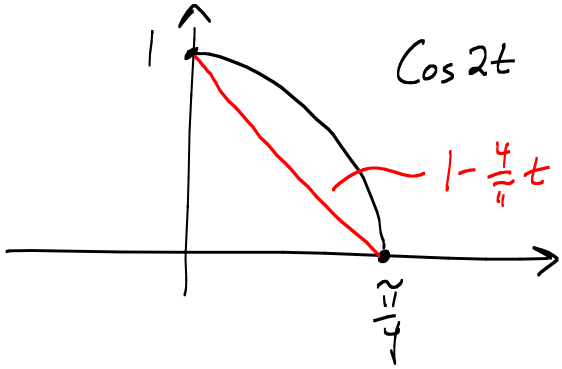
$$C_R : z(t) = Re^{it}, \quad 0 \le t \le \tfrac{\pi}{4}.$$

Show $\underbrace{\left|\int_{C_R} e^{-z^2}\,dz\right|}$ $\to 0$ as $R \to \infty$.

$$\left|\int_0^{\pi/4} e^{-(Re^{it})^2}\, iRe^{it}\,dt\right|$$

$$\le \int_0^{\pi/4} Re^{-R^2\cos 2t}\,dt$$

Hint:



$e^{-R^2(1-\frac{4}{\pi}t)}$

↑

bigger!