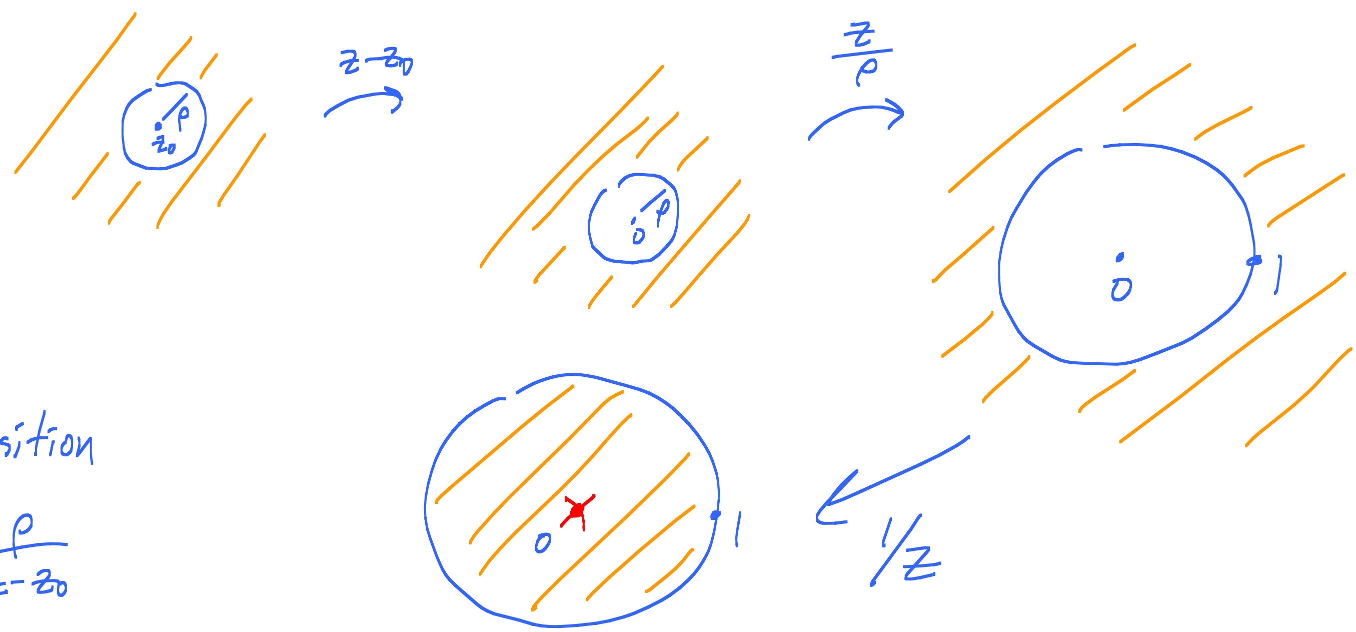
Lecture 14 Isolated singularities

Important map



Composition

$$\frac{\rho}{z - z_0}$$

Casorati–Weierstraß Thm: If an entire fcn misses an open set, then it must be constant.

Pf:

Aha! $\frac{\rho}{f(z) - z_0}$ is a bdd entire fcn. Liouville's $\Rightarrow$ const.

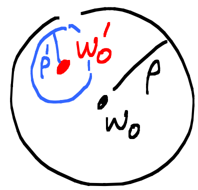
Thm: Only 3 kinds of isolated sing: 1) Removable, 2) Pole: $\lim_{z \to a} f(z) = \infty$, or 3) Essential: $f(D_\varepsilon(a) - \{a\})$ is dense in $\mathbb{C}$, no matter how small $\varepsilon > 0$.

Pf: Suppose f analytic on $D_r(a) - \{a\}$. And suppose a is not an essential sing. Then $\exists\ \varepsilon$ with $0 < \varepsilon < r$ such that

$f(D_\varepsilon(a)-\{a\})$ is not dense in $\mathbb{C}$. So $\exists\, w_0 \in \mathbb{C}$ that is not a limit pt. Let $\mathcal{A} = f(D_\varepsilon(a)-\{a\})$.

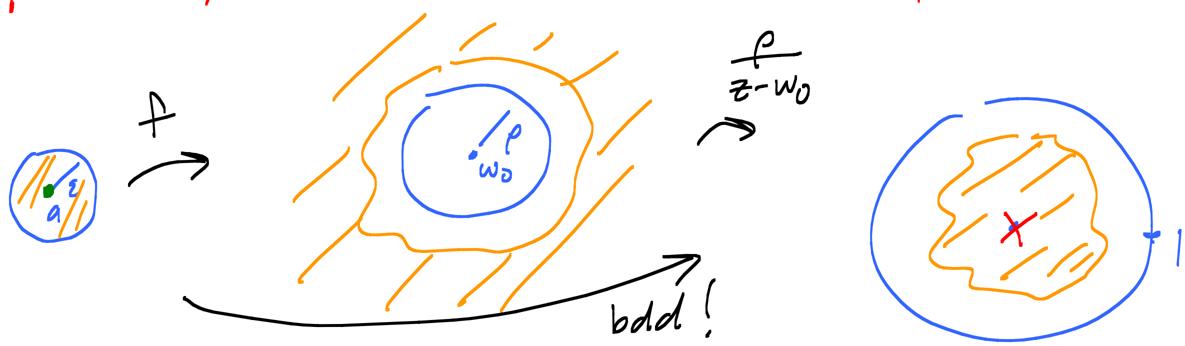
Case $w_0 \notin \mathcal{A}$ Then $\exists\, \rho > 0$ such that $\mathcal{A} \cap D_\rho(w_0) = \emptyset$.

Case $w_0 \in \mathcal{A}$:



← w_0 only point in $\mathcal{A}$ in $D_\rho(w_0)$

Replace w_0 by $w_0' \notin \mathcal{A}$. Pick ρ' so that $D_{\rho'}(w_0') \cap \mathcal{A} = \emptyset$.



Aha! Riemann Removable Sing Thm $\Longrightarrow$ a is removable for $h(z) = \dfrac{\rho}{f(z)-w_0}$. h analytic on $D_\varepsilon(a)$.

Case $h(a) \neq 0$: $f(z) = w_0 + \dfrac{\rho}{h(z)}$ ← f analytic at a!

a is removable.

Case $h(a) = 0$: See $f(z) \to \infty$ as $z \to a$

a is a pole.

Zeroes and Poles :

Zeroes: $f(z) = (z-a)^N F(z)$ where $F(a) = \dfrac{f^{(N)}(a)}{N!} \neq 0$.

a is a zero of order N.

Poles: $f(z) = (z-a)^{-N} F(z)$ where F analytic and $F(a) \neq 0$.

a is a pole of order N.

Why: Suppose a is a pole of f: $\lim_{z \to a} f(z) = \infty$.

Let $M = 1$. There is a $\delta > 0$ such $|f(z)| > 1$ on $D_\delta(a) - \{a\}$. So f non-vanishing on $D_\delta(a) - \{a\}$ and $\frac{1}{f(z)}$ is analytic on $D_\delta(a) - \{a\}$. But $\lim_{z \to a} \frac{1}{f(z)} = 0$.

Riemann removable Sing thm $\Rightarrow$ a is removable for $\frac{1}{f}$ and $\frac{1}{f}$ has a zero at a, say of order N.

$$\frac{1}{f} = (z-a)^N H(z), \quad H(a) \neq 0.$$

$$f = (z-a)^{-N} \underbrace{F(z)}_{A_0 + A_1(z-a) + \cdots} \quad \text{where} \quad F(z) = \frac{1}{H(z)}.$$

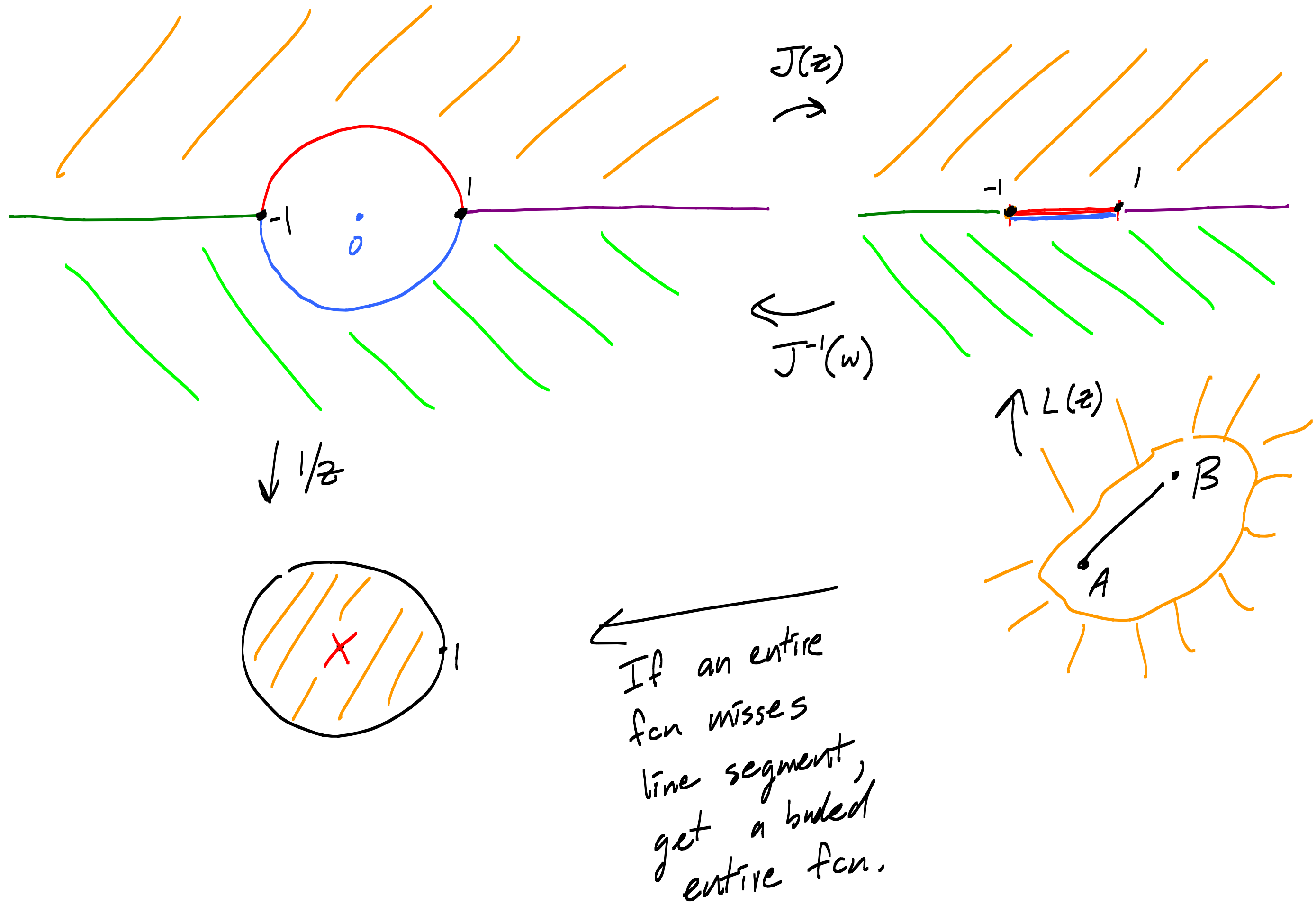
Next: Use power series:

$$f = \underbrace{\frac{A_0}{(z-a)^N} + \frac{A_1}{(z-a)^{N-1}} + \cdots + \frac{A_{N-1}}{z-a}}_{\text{Principal part of } f \text{ at } a} + \left(\begin{matrix} \text{regular} \\ \text{power} \\ \text{series} \end{matrix} \right)$$

Def^n: N is the order of the pole at a.

Fact: Princ Part $R(z)$ is uniquely determined by the condition that $f(z) - R(z)$ has a removable sing at a.

Hmmm: Famous map: Jukovsky map $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$



Conclude: If an entire fcn misses a line segment, it must be const,

Picard's little theorem: Suppose f is entire. Then, either f is a polynomial, or f assumes every value in $\mathbb{C}$ infinitely many times, with one possible exception.

Consequently, an entire fcn that misses 2 complex values must be const.

EX: e^z misses 0. $\sin z$ misses nothing.

Picard's Big thm: If f has an essential sing at a, then on any $D_\varepsilon(a)-\{a\}$ where f is analytic, f takes every value in $\mathbb{C}$ infinitely many times, with one possible exception.

EX: $e^{1/z}$ misses 0. $\sin\frac{1}{z}$ misses nothing.

Jukovsky and Cos: