Lecture 15 Partial fractions, Schwarz lemma

HWK 4 due Fri.
Midterm in class on Monday, March 4

Partial fractions: Suppose $P(z)$ and $Q(z)$ are polynomials with

$$N_Q = \deg Q > \deg P = N_P, \quad P, Q \text{ have no common factors.}$$

Step 1: Factor $Q(z) = (z-a_1)^{m_1}(z-a_2)^{m_2}\cdots(z-a_N)^{m_N}$

Near a_k:
$$\frac{P(z)}{Q(z)} = \frac{1}{(z-a_k)^{m_k}} \underbrace{\frac{P(z)}{q_k(z)}}_{\text{Expand in power series about } a_k} \leftarrow q_k(a_k) \neq 0.$$

$$\frac{P}{Q} = R_k + (\text{analytic at } a_k)$$

R_k ← principal part at a_k.

$$= \frac{A_1}{(z-a_k)^{m_k}} + \cdots + \frac{A_{m_k}}{(z-a_k)}$$

Recall: $\frac{P}{Q} - R_k$ has a removable sing at a_k.

Partial fractions decomposition: $\displaystyle \frac{P}{Q} = \sum_{k=1}^{N} R_k$

Pf: Let $\displaystyle f = \frac{P}{Q} - \sum_{k=1}^{N} R_k$.

Near a_p:
$$f = \underbrace{\left(\frac{P}{Q} - R_p\right)}_{\text{removable sing at } a_p} - \underbrace{\sum_{k \neq p} R_k}_{}$$

← analytic near a_p.

Aha! Can give f proper values at zeroes of Q to make it entire.

Claim: f is bnded. Basic Poly Estimates:

$$a|z|^{N_P} \le |P(z)| \le A|z|^{N_P}, \quad |z| > R_P$$

$$b|z|^{N_Q} \le |Q(z)| \le B|z|^{N_Q}, \quad |z| > R_Q$$

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{A}{B}\cdot\frac{1}{|z|^{\underbrace{N_Q-N_P}_{>1}}} \quad ; \text{ if } \quad |z| > \mathrm{Max}(R_P, R_Q)$$

$$\to 0 \quad \text{as} \quad |z| \to \infty$$

And $$|R_k| \le \frac{|A_1|}{|z-a_k|^{m_k}} + \cdots + \frac{|A_{m_k}|}{|z-a_k|} \to 0$$ as $|z| \to \infty$.

So $f \to 0$ as $z \to \infty$ and we see that f is bnded entire. Liouville's $\Rightarrow$ $f \equiv$ const. Constant must be zero. ✓

Freshman calculus:
$$\frac{x^3+7x+13}{(x-2)^2(x^2+x+1)} = \frac{A}{(x-2)^2} + \frac{B}{(x-2)} + \underbrace{\frac{Cx+D}{x^2+x+1}}$$

$$\frac{E}{x-r} + \frac{\bar{E}}{x-\bar{r}}$$

Schwarz lemma: (Chap. 8, sec 2)

Suppose $\underbrace{f : D_1(0) \to D_1(0)}$ is analytic and $f(0)=0$.

$|f(z)| < 1$ for z with $|z| < 1$.

Then A) $|f(z)| \le |z|$

B) $|f'(0)| \le 1$

and if equality holds in (A) for some $z \neq 0$,

or

if equality holds in (B), then

$f(z) = \lambda z$ for some unimodular constant $\lambda = e^{i\theta}$.

Pf: Let $F(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$ $\quad DQ = \frac{f(z) - \underbrace{f(0)}_{=0}}{z - 0}$

Riemann removable sing thm $\Rightarrow$ F has a removable sing at 0.

Let $z_0 \in D_1(0)$. Pick r with $0 < |z_0| < r < 1$.

Max Princ: $|F(z_0)| \le \max_{|z|=r} |F(z)| = \max_{|z|=r} \frac{|f(z)|}{r} \le \frac{1}{r}$.

Aha! Can let $r \nearrow 1$. Then $\frac{1}{r} \searrow 1$ and we conclude that $|F(z_0)| \le 1$. z_0 arbitrary.

So $\begin{cases} \left|\frac{f(z)}{z}\right| \le 1 & z \neq 0 & A \checkmark \\ |f'(0)| \le 1 & z = 0 & B \checkmark \end{cases}$

Part 2 of Schwarz follows from: If a max of $|F|$

is attained at $z_0 \in D_1(0)$, then $F \equiv c$, a constant.

$|c|$ must $= 1$. ✓

Log on a convex open set

Thm: Given a non-vanishing analytic fcn F on a convex open set Ω, there is an analytic fcn G on Ω such that $e^G = F$.

Idea: Hmmm. $\frac{d}{dz}\underbrace{e^G}_{F} = \underbrace{e^G}_{F}\cdot G'$

$$F' = F\cdot G'. \qquad \text{So} \quad G' = \frac{F'}{F}.$$

Aha! Can define G to be $G(z) = \int_{L_a^z} \frac{F'(w)}{F(w)}\,dw$ ← complex antider for $\frac{F'}{F}$

line from fixed a to z

F non-vanish!

Hope $e^G = F$. Maybe not quite. But

$$\frac{d}{dz}\left(\frac{F}{e^G}\right) = \frac{F'e^G - Fe^G G'}{(e^G)^2} \equiv 0 \qquad G' = \frac{F'}{F}$$

So $\frac{F}{e^G} \equiv c$, a const.

$$F = ce^G = e^{\mathrm{Log}\, c}\, e^G = e^{(\mathrm{Log}\, c + G)}$$

that's it!

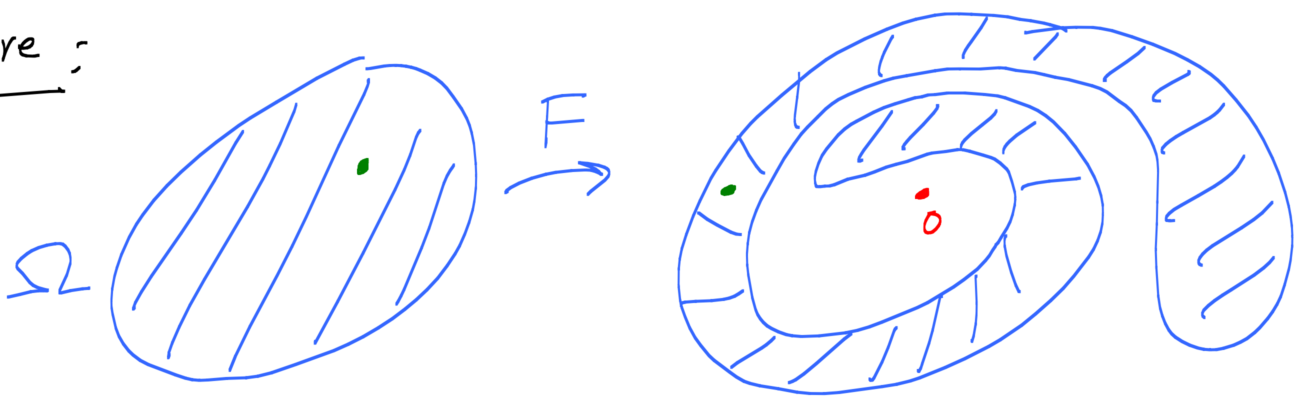
<u>Remark</u>: Can do this for Ω = simply connected domain, (no holes).

<u>Cor</u>: Same hyp. F also has an analytic N-th root on Ω.

<u>Pf</u>: $F = e^G = \left(e^{G/N}\right)^N$

N-th root.

<u>Picture</u>:



$$F = e^G$$

$G \in \log F \qquad G(z) = \mathrm{Ln}|F(z)| + i\,\theta(z)$

$\theta(z) \in \arg F(z)$.

See: possible to choose a continuous choice of $\arg F$ on Ω.

<u>Exercise</u>: Suppose H_1 and H_2 are two analytic N-th roots of F on a domain Ω.

Show that $H_1 = \lambda H_2$ on Ω where

$|\lambda| = 1$ and $\lambda^N = 1$. $\leftarrow$ λ N-th root of unity,

<u>EX</u>: $H_1^2 = H_2^2$

$$H_1^2 - H_2^2 \equiv 0$$

$$\underbrace{(H_1 - H_2)}_{\text{Eq } 1 = 0}\underbrace{(H_1 + H_2)}_{\text{Eq } 2 = 0} \equiv 0$$

Pick $z_0 \in \Omega$ and a seq $z_n \to z_0$, $z_n \neq z_0$ $\forall n$.

One of Eqns must have a seq of roots $\to z_0$.

Identity thm $\Rightarrow$ $H_1 = \pm H_2$