Lecture 18 Local mapping theorem, Conformality, toy regions

Hwk 5 due Fri.

Local Map Thm: $f(z_0)=w_0$ analytic, mult N:

$$f(z)-w_0 = (z-z_0)^N \underbrace{F(z)}_{\text{analytic and } F(z_0)\neq 0.}$$

Shrink $r>0$ so that F is analytic and non-vanishing on $\underbrace{D_r(z_0)}_{\text{convex}}$

Know $\exists$ analytic $h(z)$ on $D_r(z_0)$ such that

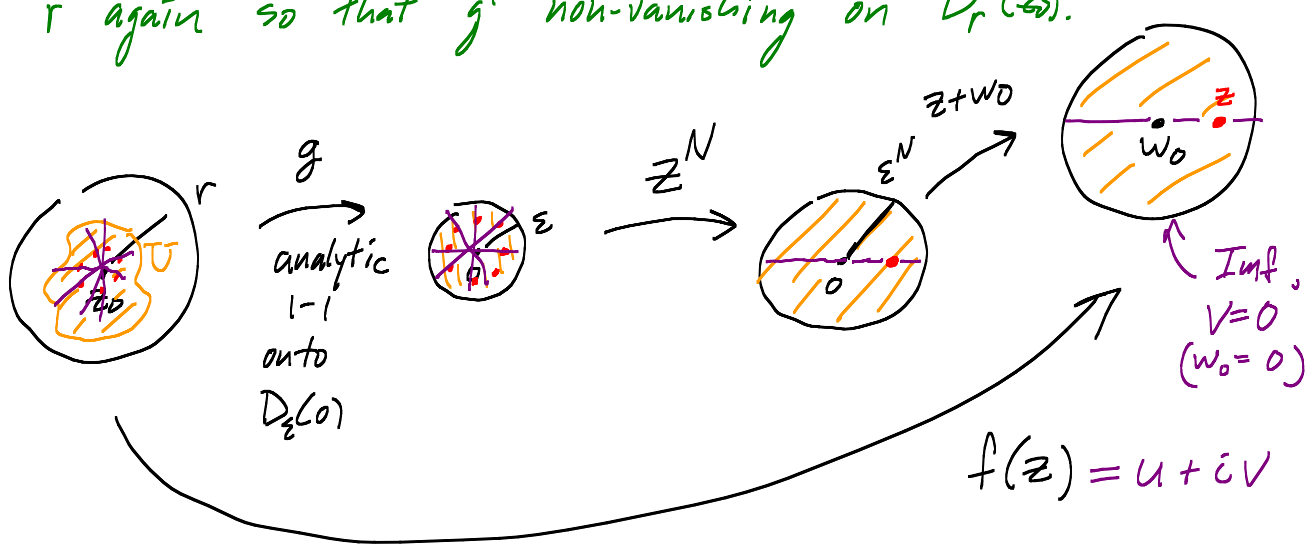
$h(z)^N = F(z)$. [h = analytic N-th root of F].

$$f(z)-w_0 = \Big[\underbrace{(z-z_0)\,h(z)}_{g(z)}\Big]^N$$

$$g'(z) = h(z) + (z-z_0)h'(z)$$

$$g'(z_0) = h(z_0) \leftarrow \text{not} = 0.$$

Shrink r again so that g' non-vanishing on $D_r(z_0)$.



See $f^{-1}(w_0)=\{z_0\}$, but $f^{-1}(w)=\{N \text{ distinct pts in } U\}$

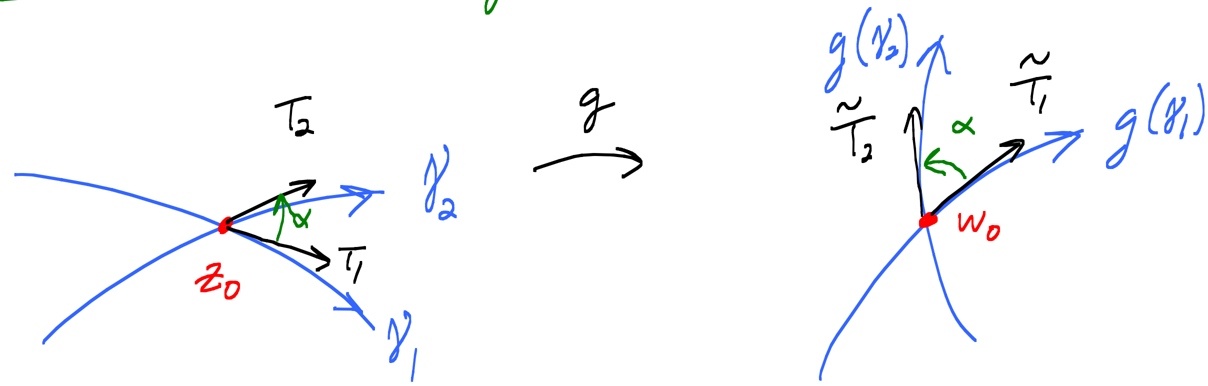
if $w\neq w_0$, $w\in D_{\varepsilon^N}(w_0)$

$f : U - \{z_0\} \to D_{\varepsilon^N}(w_0) - \{w_0\}$ is an "N-sheeted covering map."

$f : U \to D_{\varepsilon^N}(w_0)$ is a "branched covering map"

__Cor of L.M.T.__: Zeroes of non-constant harmonic fcns cannot be isolated. Given by curves that cross at angles $\frac{2\pi}{N}$.

__Big Fact__: Analytic g that is 1-1 is __conformal__.



$\gamma_j : z_j(t), \; z_j(0) = z_0$ $\qquad$ $g(\gamma_j) : g(z_j(t))$

$T_j = z_j'(0)$ $\qquad$ $\tilde{T}_j = g'(\underbrace{z_j(0)}_{z_0}) \underbrace{z_j'(0)}_{T_j}$

$$\tilde{T}_j = \underbrace{g'(z_0)}_{re^{i\theta}} T_j$$

$re^{i\theta} \leftarrow$ stretch by r, rotate by θ.

See α is pre<u>serve</u>d and the <u>sense</u> is preserved.

__Note__: g 1-1 $\Longrightarrow$ g' non-vanishing $\Longrightarrow$ conformal

Converse is (almost) true: $f(x+iy) = u(x,y) + i\,v(x,y)$

u, v $\mathcal{C}^1$-smooth and $\det\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \neq 0.$

If f is conformal, then f is analytic!

Pf: $$J = \underbrace{\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}}_{\text{causes rotation by } \theta} = \underbrace{\begin{bmatrix} R\cos\theta & -R\sin\theta \\ R\sin\theta & R\cos\theta \end{bmatrix}}_{\text{see C-R Eqns!}}$$

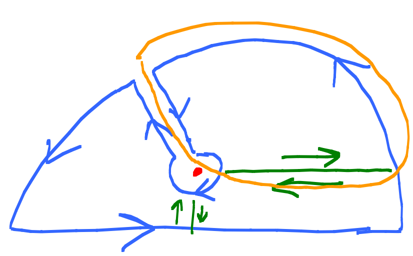
Remark: Local Map Thm $\Rightarrow$ Super inverse fcn thm!

Why: f locally 1-1 $\Rightarrow$ N must $= 1$.

$f' \neq 0$ because $g' \neq 0$. ✓

Baby residue theorem on toy regions.

Toy regions:



Convex set

$\int$  $= 0$

by Cauchy on convex

Cauchy Thm:

$\int_\gamma f\,dz = 0$

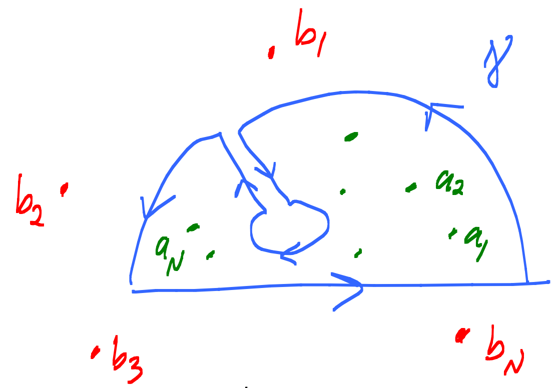
f analytic "inside and on γ"

Thm via def^n: Cauchy's Thm for toy regions is true.

Def^n: Suppose f has a pole at a with princ part

$$R(z) = \frac{A_N}{(z-a)^N} + \cdots + \frac{A_1}{z-a} \qquad A_1 = \text{Res}_a f$$

Baby Residue Thm:



h analytic on $\mathbb{C}$ except at finitely many poles. No poles on γ.

$$\int_\gamma h\,dz = 2\pi i \sum_{j=1}^{N} \text{Res}_{a_j} h = 2\pi i \sum \text{Residues of } h \text{ inside } \gamma.$$

Pf: $h - \sum_{j=1}^{N} R_j$ has removable sing at each a_j.

So $\int_\gamma h - \sum_{j=1}^{N} R_j \, dz = 0$

$$\int_\gamma h\,dz = \sum_1^N \int_\gamma R_j\,dz = \sum_1^N \int_\gamma \frac{\text{Res}_{a_j} h}{z-a_j}\,dz$$

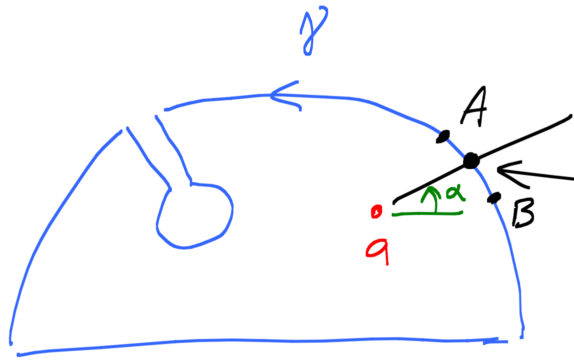
Hmmm: $\displaystyle\int_\gamma \frac{1}{(z-a)^n}\,dz = 0$ if $n \neq 1$

because $\displaystyle\frac{1}{(z-a)^n} = \frac{d}{dz}\left[\frac{1}{-n+1}\frac{1}{(z-a)^{n-1}}\right]$

Claim:
$$\int_\gamma \frac{1}{z-a}\,dz = \begin{cases} 2\pi i & a \text{ inside} \\ 0 & a \text{ outside} \end{cases}$$

by Cauchy's.

Pf:



only cuts at one point with an angle.

γ_A^B

$$\int_\gamma \frac{1}{z-a}\,dz = \lim_{\substack{A \text{ slides down} \\ B \text{ slides up}}} \int_{\gamma_A^B} \underbrace{\frac{1}{z-a}}_{\frac{d}{dz}\log_\alpha(z-a)}\,dz$$

$$= \log_\alpha(B-a) - \log_\alpha(A-a)$$

$$= \Big(\operatorname{Ln}|B-a| + i\,(\alpha + 2\pi - \varepsilon)\Big) - \Big(\operatorname{Ln}|A-a| + i\,(\alpha + \tilde{\varepsilon})\Big)$$

$$\longrightarrow i\,2\pi \quad \text{as } A\downarrow,\ B\nearrow$$

Now let $H(w) = \int_\gamma \frac{1}{z-w}\,dz$. Know $H(a) = 2\pi i$

$$H'(w) = \int_\gamma \frac{1}{(z-w)^2}\,dz = \int_\gamma \frac{d}{dz}\left[\frac{-1}{z-w}\right]dz \equiv 0$$

Aha! $H(w) \equiv 2\pi i$ inside γ.