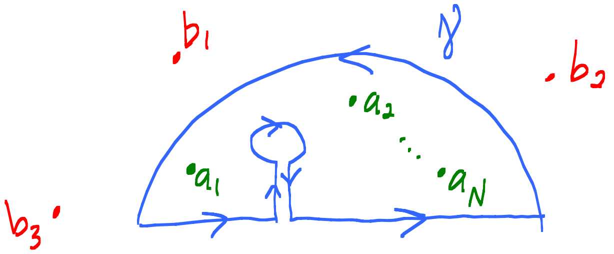
Lecture 19 Residue theorem

Practice problems and old Exam 1 on Home page now. Exam 1 in class Monday, March 4



$$\int_\gamma f\,dz = 2\pi i \sum_{n=1}^{N} \operatorname{Res}_{a_n} f$$

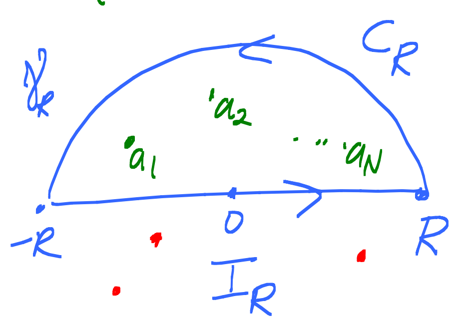
Theorem: Suppose $P(z)$ and $Q(z)$ are polynomials and

1) $\deg Q \geq \deg P + 2$
2) Q has no zeroes on $\mathbb{R}$

Then $$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \frac{P}{Q}$$

where $Z_Q^+ = \{\text{zeroes of } Q \text{ in UHP}\}$ ← upper half plane

Pf:



$I_R: \; z(t) = t$

$-R \leq t \leq R$

$z'(t) = 1$

So $$\int_{I_R} \frac{P}{Q}\,dz = \int_{-R}^{R} \frac{P(t)}{Q(t)} \cdot 1 \cdot dt$$ ← want as $R \to \infty$

Make $R > 0$ large enough so that all $a \in Z_Q^+$ are inside γ_R

$$\int_\gamma \frac{P}{Q}\,dz = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \frac{P}{Q}$$ ← S. of R.'s

Claim: $$\int_{C_R} \frac{P}{Q}\,dz \to 0 \text{ as } R \to \infty.$$

Key: Basic poly estimate:

$$a|z|^{\deg P} \le \underline{|P(z)| \le A|z|^{\deg P}} \text{ if } |z|>R_P$$

$$\underline{b|z|^{\deg Q} \le |Q(z)|} \le B|z|^{\deg Q} \text{ if } |z|>R_Q$$

$$\left|\frac{P}{Q}\right| \le \frac{A}{b} \frac{1}{|z|^{\deg Q - \deg P}} \le \frac{A}{b}\frac{1}{|z|^2}$$

shows $\frac{P}{Q}$ abs. integ. on $\mathbb{R}$.

$$\text{if } |z| > \text{Max}(R_P, R_Q, 1)$$

Aha! $$\left|\int_{C_R} \frac{P}{Q}\,dz\right| \le \left(\text{Max}_{C_R} \left|\frac{P}{Q}\right|\right) \cdot \text{Length}(C_R)$$

$$\le \frac{A}{b}\frac{1}{R^2}\cdot(\pi R) \to 0 \text{ as } R\to\infty.$$

$$\int_{\gamma_R} = \int_{I_R} + \int_{C_R} = 2\pi i\,(\text{S. of R's})$$

Let $R\to\infty$ $$\int_{-\infty}^{\infty} \frac{P}{Q}\,dx + 0 = \checkmark$$

EX: $$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$$

$$z^4+1=0$$

$$z = \underline{e^{i\pi/4}, e^{i3\pi/4}}, e^{i5\pi/4}, e^{i7\pi/4}$$

UHP

$$= 2\pi i\left(\text{Res}_{e^{i\pi/4}} \frac{1}{z^4+1} + \text{Res}_{e^{i3\pi/4}} \frac{1}{z^4+1}\right)$$

Hmmm: $\displaystyle \frac{1}{z^4+1} = \frac{1}{(z-r_1)} \cdot \Bigg[\underbrace{\frac{1}{(z-r_2)(z-r_3)(z-r_4)}}_{A_0 + A_1(z-r_1) + \cdots} \Bigg]$

$$= \frac{A_0}{z-r_1} + \underbrace{A_1 + A_2(z-r_1) + \cdots}_{\substack{\text{power series} \\ \text{analytic at } r_1}}$$

Res_{r_1}

So $\displaystyle A_0 = \frac{1}{(r_1-r_2)(r_1-r_3)(r_1-r_3)} \longleftarrow$ Ugh!

<u>Fact</u>: Suppose f and g are analytic near a and g has a <u>simple</u> zero at a [meaning that $g(a)=0$, $g'(a) \neq 0$].

Then $\displaystyle \boxed{\text{Res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}}$.

<u>Why</u>: $g(z) = a_0 + a_1(z-a) + \cdots = (z-a)\big[\underbrace{a_1 + a_2(z-a) + \cdots}_{G(z)}\big]$

$g(a)=0$ $\quad \frac{g'(a)}{1!} \neq 0$

$$G(a) = a_1 = \frac{g'(a)}{1!}$$

$$\frac{f(z)}{g(z)} = \frac{1}{z-a}\Bigg[\underbrace{\frac{f(z)}{G(z)}}_{A_0 + A_1(z-a) + \cdots}\Bigg]$$

$$= \frac{A_0}{z-a} + \text{power series}$$

Res_a

So $\displaystyle \text{Res}_a \frac{f}{g} = A_0 = \frac{f(a)}{G(a)} = \frac{f(a)}{g'(a)}$ ✓

Now $\dfrac{1 \leftarrow f(z)}{z^4+1 \leftarrow g(z)=z^4+1}$, $\quad g'(z)=4z^3$

$g(e^{i\pi/4})=0$ ✓

$\underbrace{g'(e^{i\pi/4}) = 4(e^{i\pi/4})^3 \neq 0}_{g \text{ has a simple zero at } e^{i\pi/4}}$

$$\operatorname{Res}_{e^{i\pi/4}} \frac{1}{z^4+1} = \frac{1}{4e^{i3\pi/4}} = \frac{1}{4}e^{-i3\pi/4}$$

$$\int_{-\infty}^{\infty} \frac{1}{t^4+1}\,dt = 2\pi i\left[\frac{1}{4e^{i3\pi/4}} + \frac{1}{4(e^{i3\pi/4})^3}\right]$$

$$= \frac{\pi i}{2}\left[\underbrace{e^{-i3\pi/4}}_{-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}} + \underbrace{e^{-i9\pi/4}}_{\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}}\right] = \frac{\pi}{2}\,\frac{2}{\sqrt{2}} = \boxed{\frac{\pi}{\sqrt{2}}}$$

Fourier Transform:

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

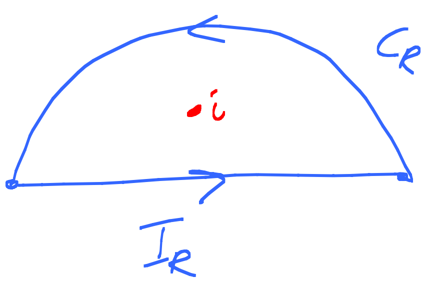
EX:

$$I(s) = \int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+1}\,dx \;\leftarrow \text{assume } s>0$$

$$= 2\pi i \operatorname{Res}_i \frac{e^{isz}}{z^2+1} \leftarrow \text{simple zero at } i$$

$$= 2\pi i \cdot \frac{e^{is(i)}}{2i}$$

$$= \pi e^{-s}$$



Works: $\quad e^{isz} = e^{is(x+iy)} = \underset{\substack{\uparrow \\ \text{modulus} \\ =1}}{e^{isx}}\; e^{-sy} \leftarrow \le 1 \text{ if } y \ge 0$

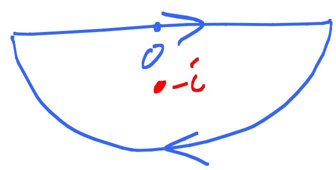
Key: $|e^{isz}| \le 1$ if $\operatorname{Im} z \ge 0$.

Denom estimate $|z^2+1| \ge \big|\,|z|^2 - |-1|\,\big| = R^2 - 1$

if $|z| = R > \underline{\underline{1}}$

So $\left| \int_{C_R} \frac{e^{isz}}{z^2+1}\, dz \right| \le \frac{1}{R^2-1} \cdot \pi R \to 0$ as $R \to \infty$.

<u>What if $s < 0$?</u>



clockwise!

Aha! $|e^{isz}| \le 1$ if $\operatorname{Im} z \le 0$

$$\int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+1}\, dx = -2\pi i \operatorname{Res}_{-i} \frac{e^{isz}}{z^2+1}$$