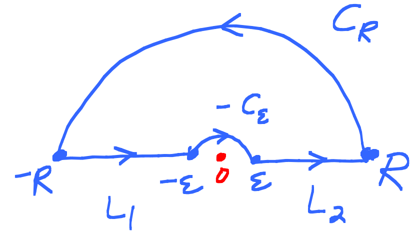
Lecture 20

$$I = \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$$



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Hmmm: $g(z) = \dfrac{\sin z}{z}$ Nope!

Correct: $f(z) = \dfrac{e^{iz}}{z}$ $\qquad f(t) = \dfrac{e^{it}}{t} = \underbrace{\dfrac{\cos t}{t}}_{\text{odd}} + i\underbrace{\dfrac{\sin t}{t}}_{\text{even}}$

$$\left(\int_{L_1} + \int_{L_2}\right) f(z)\,dz = 0 + i\,2\int_\varepsilon^R \frac{\sin t}{t}\,dt \leftarrow \text{want!}$$

On $-C_\varepsilon$: $\quad \dfrac{e^{iz}}{z} = \dfrac{1 + (iz) + \frac{(iz)^2}{2!} + \cdots}{z} = \dfrac{1}{z} + \underbrace{\left(i - \dfrac{z}{2!} + \cdots\right)}_{H(z)\text{ entire}}$

$\uparrow$ simple pole at $z=0$

$H(z)$ entire so bdd by M on $\overline{D_1(0)}$.

$$\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz = \underbrace{\int_{-C_\varepsilon} \frac{1}{z}\,dz}_{} + \underbrace{\int_{-C_\varepsilon} H\,dz}_{}$$

$$-\int_{C_\varepsilon} \frac{1}{z}\,dz = -\int_0^\pi \frac{1}{\varepsilon e^{it}}\left(i\varepsilon e^{it}\right)dt = -i\pi$$

$$\left|\ \cdots\ \right| \le \left(\max_{C_\varepsilon} |H|\right)\pi\varepsilon \le M\pi\varepsilon \text{ if } \varepsilon < 1.$$

So $\displaystyle\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz \to -i\pi$ as $\varepsilon \to 0$.

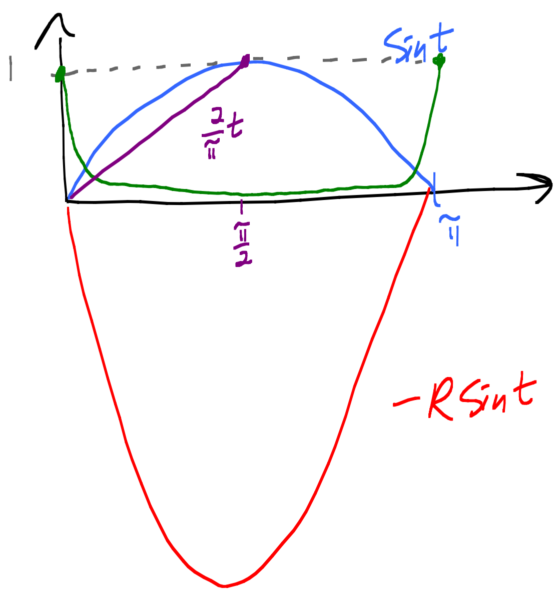
On C_R: $\quad \left|\displaystyle\int_{C_R} \frac{e^{iz}}{z}\,dz\right| \le \dfrac{\max_{C_R} |e^{iz}|}{R} \cdot \pi R \le \pi$ $\qquad \max_{C_R} |e^{iz}| = 1$ $\qquad$ Ouch!

Basic estimate fails us.

Jordan's Lemma:

$$\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| = \left|\int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}}\, iRe^{it}\,dt\right|$$

$$= \left|\int_0^{\pi} i\,e^{iR\cos t - R\sin t}\,dt\right| \le \int_0^{\pi} e^{-R\sin t}\,dt$$



$0 \le \frac{2}{\pi}t \le \sin t$ on $[0, \frac{\pi}{2}]$

$e^{-R\sin t} \le e^{-R\frac{2}{\pi}t}$ on $[0, \frac{\pi}{2}]$

$$\int_0^{\pi} e^{-R\sin t}\,dt$$

$$\underset{\text{by symmetry}}{=} 2\int_0^{\pi/2} e^{-R\sin t}\,dt$$

$$\le 2\int_0^{\pi/2} e^{-R\frac{2}{\pi}t}\,dt$$

$$= 2\left[-\frac{\pi}{2R}\,e^{-R\frac{2}{\pi}t}\right]_0^{\pi/2}$$

$$= \frac{\pi}{R}\left(1-e^{-R}\right)$$

$$< \frac{\pi}{R} \to 0 \text{ as } R\to\infty$$

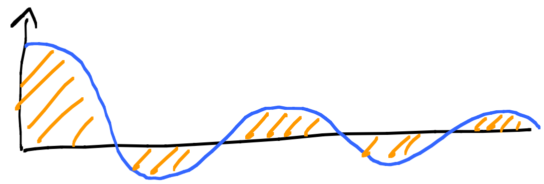
Last step: $\left(\int_{L_1} + \int_{L_2} + \int_{-C_\varepsilon} + \int_{C_R} \right) \frac{e^{iz}}{z}\, dz = 0$

$$\underbrace{\int_{L_1} + \int_{L_2}}_{2i\int_\varepsilon^R \frac{\sin t}{t}\,dt} + \int_{-C_\varepsilon} + \int_{C_R} \quad\to\quad 2i\int_\varepsilon^R \frac{\sin t}{t}\,dt \; - i\pi \; + \; 0 \; = 0$$

Cauchy on Toys

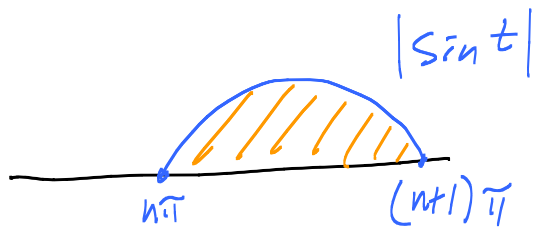
Let $\varepsilon \to 0$. Then $R \to \infty$. Get $I = \frac{\pi}{2}$. ✓

$\int_0^\infty \frac{\sin t}{t}\, dt$ is famous:



Alternating series test shows $\int_0^R \longrightarrow$ converges

But $\int_0^\infty \left| \frac{\sin t}{t} \right| dt = \infty$



Area = 2

$\int_{n\pi}^{(n+1)\pi}$: $\frac{|\sin t|}{(n+1)\pi} \le \frac{|\sin t|}{t} \le \frac{|\sin t|}{\pi n}$ for $t \in [n\pi, (n+1)\pi]$

$$\frac{2}{(n+1)\pi} \le \int_{n\pi}^{(n+1)\pi} \left| \frac{\sin t}{t} \right| dt \le \frac{2}{\pi n}$$

shows $\int_0^\infty \frac{\sin t}{t}\, dt$ blows up like harmonic series $\sum^N \frac{1}{n}$

$\sim \ln N$

Theorem: P, Q polynomials. Q has no zeroes on $\mathbb{R}$.

$\deg Q \geq \deg P + 1$ ← instead of 2

Then
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ix}\, dx = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \left(\frac{P(z)}{Q(z)} e^{iz} \right)$$

where Z_Q^+ are the zeroes of Q in UHP.

Pf: Basic est. fails on C_R, but Jordan's Lemma saves the day.

EX: $\displaystyle\int_0^{2\pi} \sin^4\theta \, d\theta$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{e^{i\theta} - \frac{1}{e^{i\theta}}}{2i}$$

Aha! $C: z(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$

$z'(\theta) = ie^{i\theta}$

$$\int_0^{2\pi} \left(\frac{z(\theta) - \frac{1}{z(\theta)}}{2i} \right)^4 \underbrace{\frac{ie^{i\theta}}{ie^{i\theta}}\, d\theta}_{dz}$$

$iz(\theta)$

$$= \int_C \left(\frac{z - \frac{1}{z}}{2i} \right)^4 \frac{1}{iz}\, dz = \frac{1}{16} \int_{C_1} \underline{\left(z^4 - 4z^2 + 6 - 4\frac{1}{z^2} + \frac{1}{z^4} \right) \frac{1}{iz}}\, dz$$

Residue term!

$$= 2\pi i \cdot \frac{1}{16} \operatorname{Res}_0 \left[\underline{\qquad} \right] = 2\pi i \cdot \frac{1}{16} \cdot \left(\frac{6}{i} \right) = \frac{3\pi}{4}$$

Method: $R(z,w)$ rational in z and w.

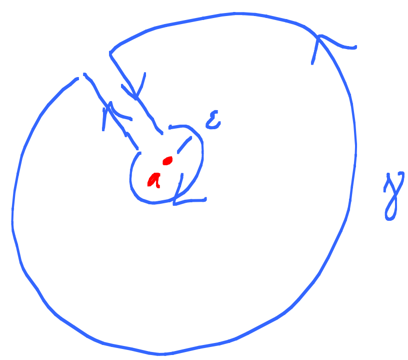
no blowing up on $C_1(0)$.

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

$$= \int_C \underbrace{R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\cdot\frac{1}{iz}}_{H(z)}\, dz$$

$$= 2\pi i \sum_{\substack{\text{poles} \\ \text{inside } C_1(0)}} \operatorname{Res}_a H(z)$$

Stein: Key hole arguments



Cauchy's $\displaystyle\int_\gamma \frac{f(z)}{z-a}\, dz = 0$

Close sides and let $\varepsilon \to 0$.

$$\int_{\text{outer curve}} \frac{f(z)}{z-a}\, dz = \lim_{\varepsilon\to 0} \underbrace{\int_{C_\varepsilon} \frac{f(z)}{z-a}\, dz}_{\int_0^{2\pi} \frac{f(a+\varepsilon e^{it})}{(a+\varepsilon e^{it})-a}\, i\varepsilon e^{it}\, dt} = 2\pi i\, f(a)$$

$$= i\int_0^{2\pi} f(a+\varepsilon e^{it})\, dt$$

If f is continuous at a.