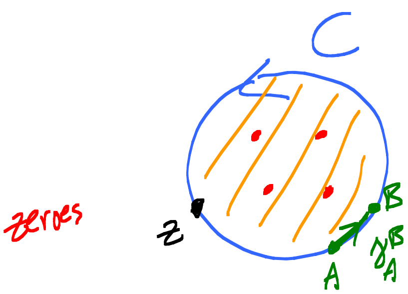
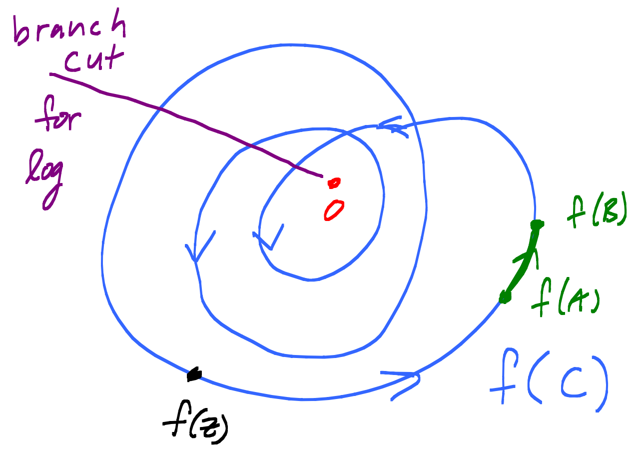
Lecture 21 Residue integrals

Why "argument principle"? $\displaystyle \frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \left(\begin{array}{c}\#\text{ zeroes of } f\\ \text{inside } C\end{array}\right)$



f non-vanishing on C
analytic inside and on



$$\int_{\gamma_A^B} \frac{f'}{f}\,dz = \int_{\gamma_A^B} \frac{d}{dz}\Big[\log f\Big]\,dz$$

$$= \log f(B) - \log f(A)$$

$$= \Big(\operatorname{Ln}|f(B)| + i\arg f(B)\Big) - \Big(\operatorname{Ln}|f(A)| + i\arg f(A)\Big)$$

$$= \Big(\operatorname{Ln}|f(B)| - \operatorname{Ln}|f(A)|\Big) + i\,\Delta_{\gamma_A^B}\arg f$$

Add up segments going around C. $\operatorname{Ln}|f|$ terms pairwise cancel and the first cancels the last!

$$\boxed{\int_C \frac{f'}{f}\,dz = i\sum \Delta_{\gamma_{A_i}^{B_i}}\arg f = i\,\Delta_C \arg f}$$

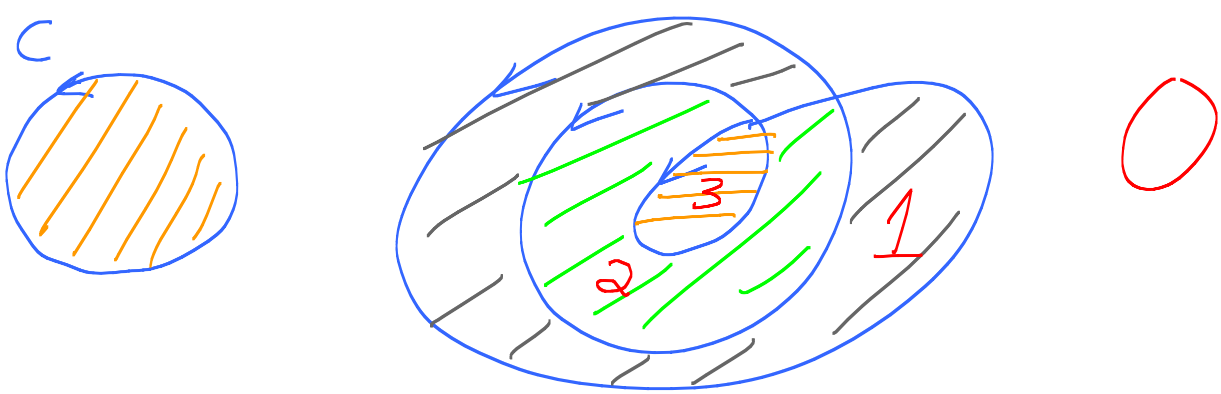
Aha! $\displaystyle \frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \left(\begin{array}{c}\#\text{ times } f(z) \text{ spins}\\ \text{around origin as } z \text{ goes around } C\end{array}\right)$

Note: $$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-w}\,dz = \left(\begin{array}{c}\#\text{ times } f(z)=w \\ \text{in } C \\ \text{counted with mult.}\end{array}\right)$$

$$= \underbrace{\frac{1}{2\pi i}\int_{f(C)} \frac{1}{\zeta - w}\,d\zeta}_{H(w)} \longleftarrow$$ winding # of $f(C)$ about w.

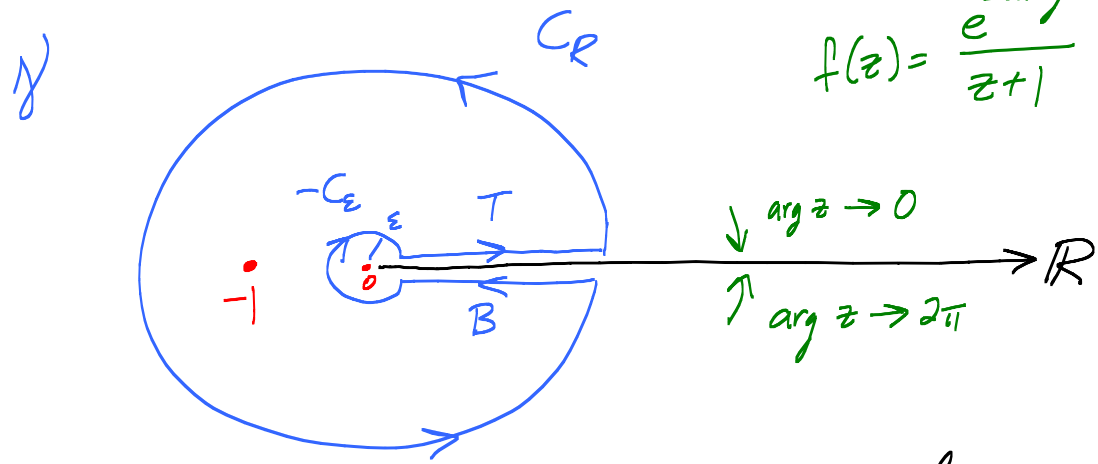
Last time: $H'(w) \equiv 0$. So $H \equiv$ const on connected components of $\mathbb{C} - f(C)$.

$$\left[\frac{d}{dw}\int \frac{1}{\zeta - w}\,d\zeta = \int \frac{1}{(\zeta-w)^2}\,d\zeta = \int \frac{d}{d\zeta}\left[\frac{-1}{\zeta - w}\right]d\zeta = 0\right]$$



EX: $$I = \int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx \qquad 0<\alpha<1$$

$$x^{-\alpha} = e^{-\alpha \ln x} \qquad z^{-\alpha} = e^{-\alpha \log z} \longleftarrow$$ branch?



$$f(z) = \frac{e^{-\alpha \log z}}{z+1}$$

Residue Thm $\displaystyle\int_\gamma \frac{e^{-\alpha \log z}}{z+1}\,dz = 2\pi i \operatorname{Res}_{-1} \frac{e^{-\alpha \log z}}{z+1}$

$$= 2\pi i \quad e^{-\alpha \log(-1)}$$

$$= 2\pi i \quad e^{-\alpha\left(\underbrace{\operatorname{Ln}|-1|}_{0} + i\pi\right)}$$

$$= 2\pi i \; e^{-i\alpha\pi}$$

Take limit as "mouth" closes:

$$\int_T \longrightarrow \int_\varepsilon^R \frac{e^{-\alpha(\operatorname{Ln} t + i\,0)}}{t+1}\,dt \quad \longleftarrow \text{want!}$$

$$\int_B \longrightarrow -\int_\varepsilon^R \frac{e^{-\alpha(\operatorname{Ln} t + i\,2\pi)}}{t+1}\,dt$$

$$= -e^{-i\alpha 2\pi} \int_\varepsilon^R \frac{t^{-\alpha}}{t+1}\,dt$$

Next,

$$\left|e^{-\alpha \log Re^{it}}\right| = \left|e^{-\alpha(\operatorname{Ln} R + it)}\right| = e^{-\alpha \operatorname{Ln} R} = R^{-\alpha}$$

$$|z+1| = |z-(-1)| \geq \big||z|-1\big| \geq R-1 \quad \text{if } |z|=R>1.$$

So $$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^{-\alpha}}{R-1}\cdot(2\pi R) \longrightarrow 0$$ as $R\to\infty$. because $\alpha>0$

but $|z+1| \ge 1-\varepsilon$ if $|z|=\varepsilon<1$.

So $$\left|\int_{C_\varepsilon} f\,dz\right| \le \frac{\varepsilon^{-\alpha}}{1-\varepsilon}\cdot(2\pi\varepsilon) = 2\pi\,\frac{\varepsilon^{1-\alpha}}{1-\varepsilon} \longrightarrow 0$$ because $\alpha<1$

Finally $$\underbrace{\int_T + \int_B}_{\to\ (1-e^{-i\alpha 2\pi})\int_\varepsilon^R \frac{t^{-\alpha}}{t+1}\,dt} + \underbrace{\int_{C_R}}_{\to\ 0} + \underbrace{\int_{-C_\varepsilon}}_{\to\ 0} = 2\pi i\, e^{-i\alpha\pi}$$

$$\begin{array}{c} T \\ \downarrow \\ \hline \uparrow \\ B \end{array}$$

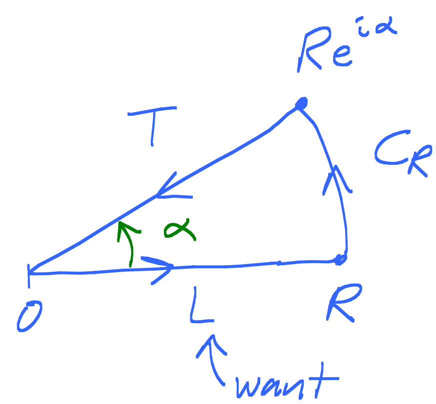
Next. Let $R\to\infty$.

Then let $\varepsilon\to 0$

Get $$I = \frac{2\pi i\, e^{-i\alpha\pi}}{1-e^{-i\alpha 2\pi}}\;\frac{e^{i\alpha\pi}}{e^{i\alpha\pi}} = \pi\,\frac{2i}{e^{i\alpha\pi}-e^{-i\alpha\pi}}\cdot 1$$

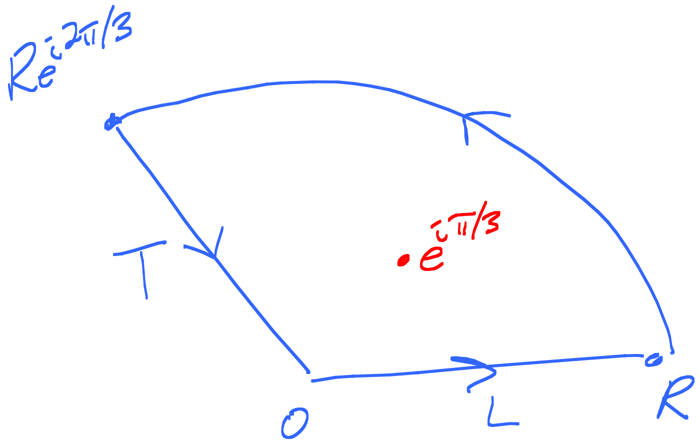
$$= \frac{\pi}{\sin\alpha\pi}$$

Next time: $$\int_0^\infty \frac{1}{x^3+1}\,dx$$



$$-T: \ z(t) = te^{i\alpha}, \quad 0 \le t \le R$$

$$\int_T \frac{1}{z^3+1}\, dz = -\int_{-T} = -\int_0^R \frac{1}{\underbrace{(te^{i\alpha})^3}_{t^3 e^{i3\alpha}}+1} \underbrace{\left(e^{i\alpha}\right)}_{z'(t)} dt$$

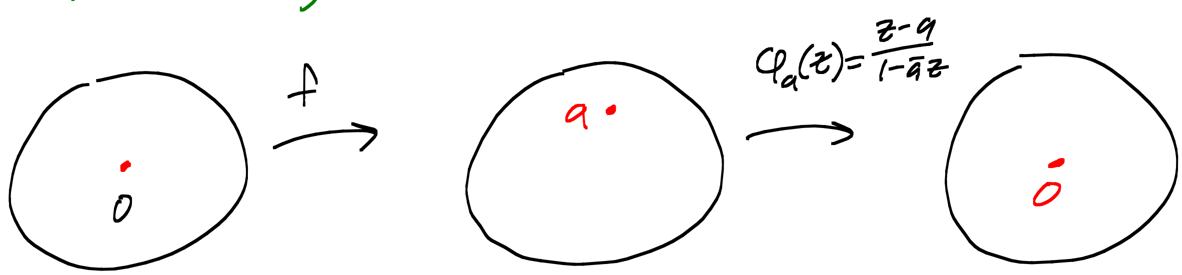


Aha! Need $e^{i3\alpha} = 1$

Take $\alpha = \frac{2\pi}{3}$

Famous Schwarz lemma Cor: $f: D_1(0) \to D_1(0)$ analytic, but don't assume $f(0)=0$. Still true that $|f'(0)| \le 1$. And if $|f'(0)| = 1$, then $f(z) = \lambda z$, $|\lambda| = 1$.

Hint:



Schwarz on $\varphi_a(f(z))$