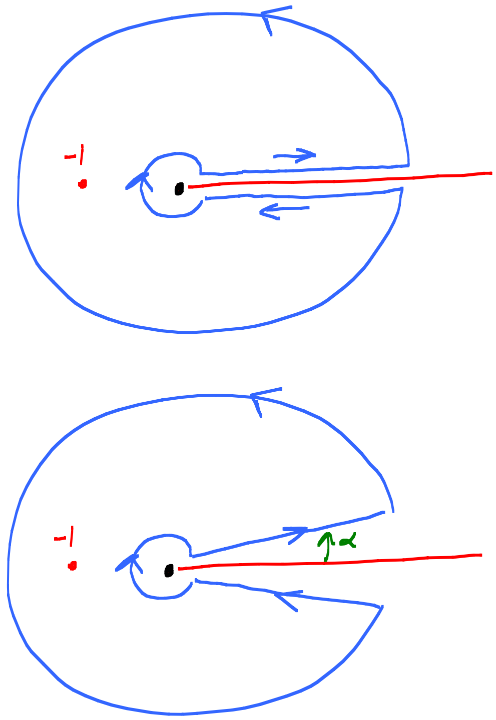
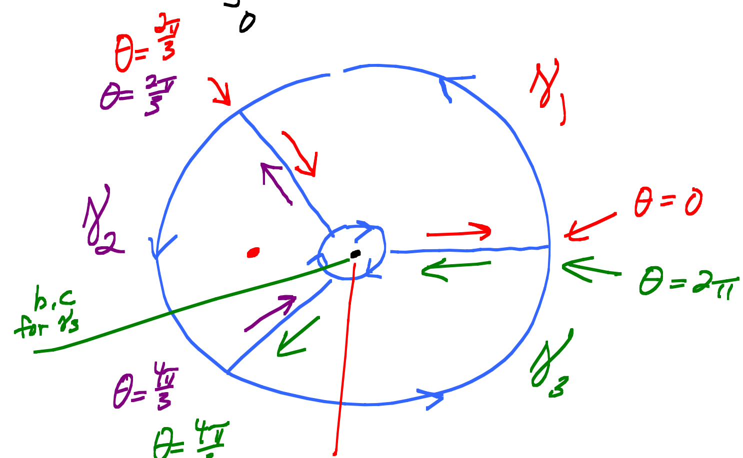
Lecture 22 Schwarz reflection

HWK 6 due Friday after break

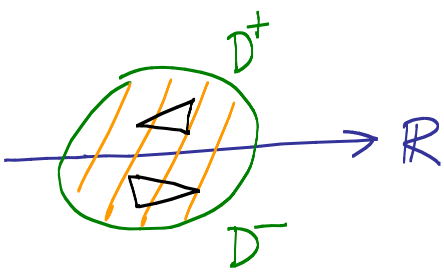
Remark:



$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx, \quad 0<\alpha<1$$



Lemma:

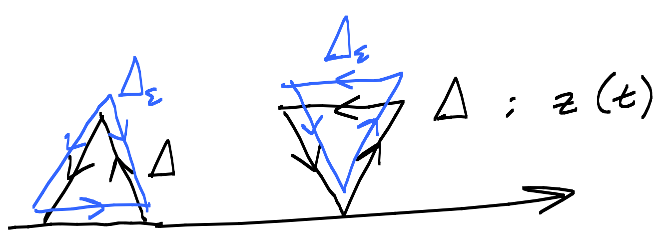


f continuous on D, analytic on D^+ and D^-. open

Then f analytic on D.

Pf: Δ in D^+ or D^-

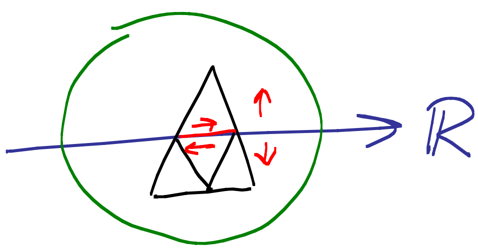
Then $\int_\Delta f\,dz = 0$



$\Delta : z(t)$

$\Delta_\varepsilon : z(t) + i\varepsilon$

$$0 = \int_{\Delta_\varepsilon} f\,dz \longrightarrow \int_\Delta f\,dz$$

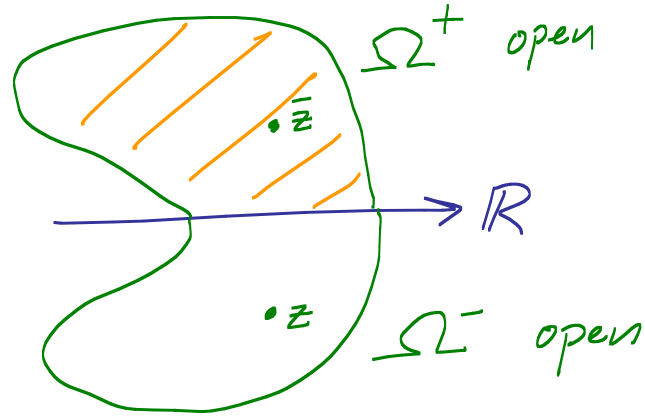


Let $\varepsilon \to 0$. See $\int_\Delta f\,dz$.

Morera's Thm $\Rightarrow$ f analytic.

Schwarz Reflection Principle

Ω symmetric about $\mathbb{R}$



1) f continuous on Ω^+ up to $\mathbb{R}$, and
2) $f(x) \in \mathbb{R}$ for $x \in \Omega \cap \mathbb{R}$, and
3) f is analytic on Ω^+.

Then
$$F(z) = \begin{cases} f(z) & z \in \Omega^+ \\ f(x) & x \in \mathbb{R} \\ \overline{f(\bar{z})} & z \in \Omega^- \end{cases}$$

is analytic on Ω.

Pf: HWK prob: $\overline{f(\bar{z})}$ is analytic on Ω^-.

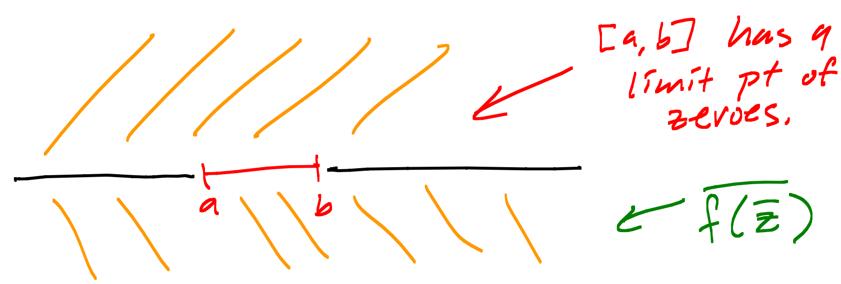
$f: \mathbb{R} \to \mathbb{R}$ means $\overline{f(\bar{z})}$ extends continuously up to $\mathbb{R}$.

Lemma $\Rightarrow$ F analytic. $\left[\overline{f(\bar{x})} = f(x), \; x \in \mathbb{R}.\right]$

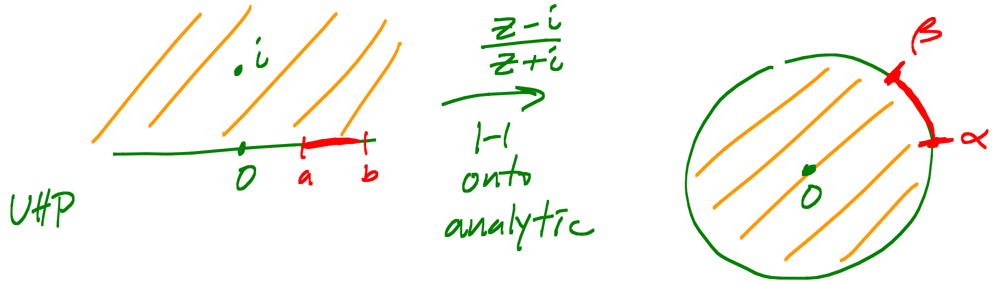
Application: Suppose f is analytic on UHP and continuous up to $[a,b] \subset \mathbb{R}$ and $f(x) = 0$ for $a < x < b$.

Then $f \equiv 0$.

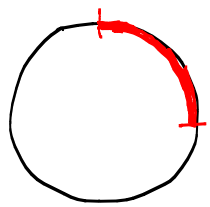
Pf: UHP $= \Omega^+$



Important tool:



Remark: Standard excersize: Suppose f continuous on $\overline{D_1(0)}$ and analytic on $D_1(0)$ and $f(e^{i\theta})=0$ for $0 \le \theta \le \frac{\pi}{2}$.



Hmmm. $F(z) = f(z)f(iz)f(-z)f(-iz)$

is continuous on $\overline{D_1(0)}$,

analytic on $D_1(0)$, $\equiv 0$ on $C_1(0)$.

Max Princ $\Longrightarrow$ $F \equiv 0$.

Exam prob $\Longrightarrow$ at least one factor $\equiv 0$.

$\Longrightarrow f \equiv 0$.

Linear Fractional Transformations (LFTs)

$$L(z) = \frac{az+b}{cz+d} \qquad (ad-bc) \neq 0$$

> **Remark:** Möbius Transf: $\frac{z-a}{1-\bar{a}z}$ is an LFT

Note: $ad-bc=0$, then top and bottom are lin dependent, so one would be a multiple of the other. LFT $\equiv$ const or worse! if $c=d=0$.

Lemma: LFTs are composed of 4 basic maps

1) $z \mapsto rz$, $r \in \mathbb{R}^+$ (stretch)

2) $z \mapsto e^{i\theta} z$ (rotation)

3) $z \mapsto z + b$ (translation)

4) $z \mapsto \frac{1}{z}$ (inversion)

<u>Pf</u>: $c = 0$ case: $L(z) = Az + B$ ✓ 1, 2, 3 enough

$re^{i\theta} = A$

$c \neq 0$: $$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{\left(b - \frac{ad}{c}\right)}{cz+d}$$

$\left(b - \frac{ad}{c}\right) = Re^{i\varphi}$

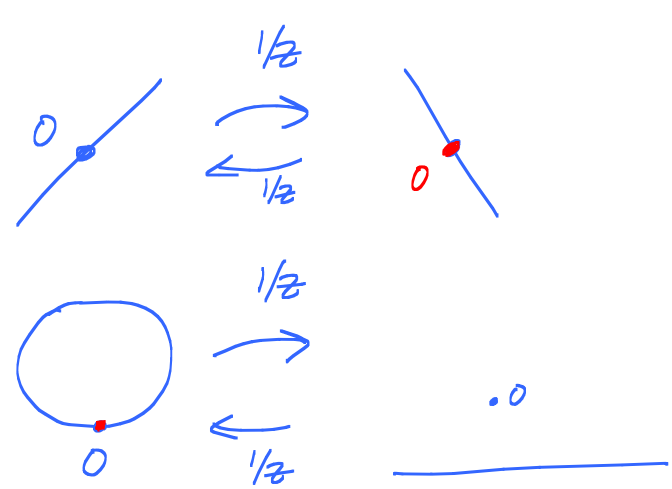
$cz + d = re^{i\theta}$

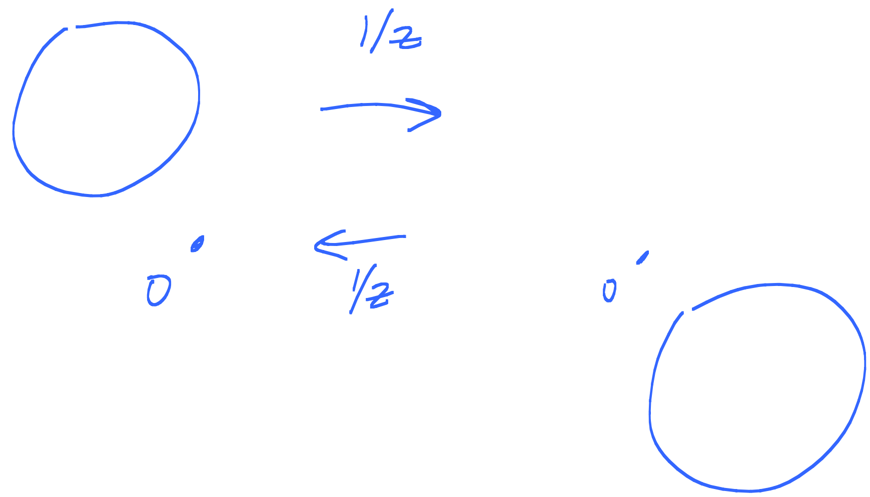
Need 7 of the basic maps!

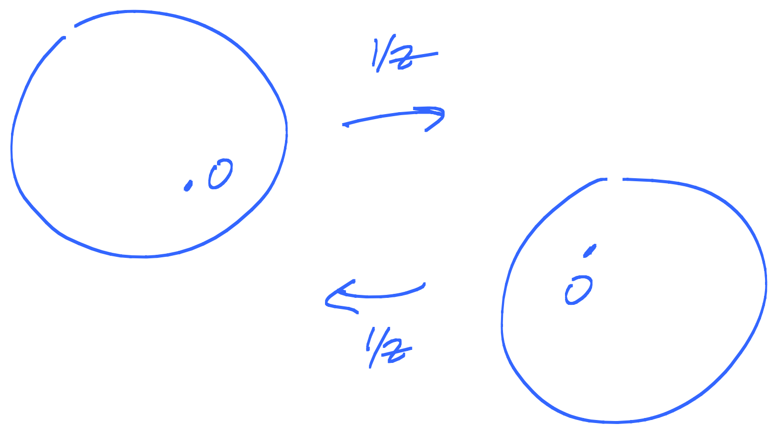
<u>Cor</u>: LFTs take {lines, circles}

<u>Pf</u>: 1, 2, 3 take {lines}, {circles}

4 Inversion mixes them up:




1/2
1/2


1/2
1/2