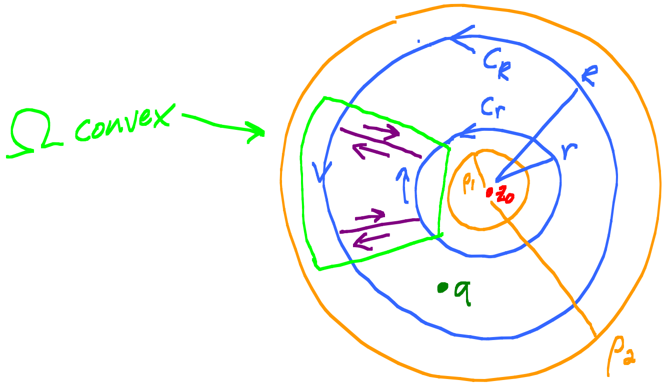
Lecture 25 Laurent expansions

HWK 6 due Friday



Suppose f analytic on

$$A(\rho_1, \rho_2) = \{z : \rho_1 < |z - z_0| < \rho_2\}$$

$$\rho_1 < r < R < \rho_2$$

Cauchy Theorem for an annulus: $$\left(\int_{C_R} - \int_{C_r}\right) f\, dz = 0$$

Pf: Add up "pieces of pie". Use Cauchy on convex. Cancel out cuts.

Cauchy Integral Formula:

$$f(a) = \frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right)\frac{f(z)}{z-a}\,dz$$

Pf: Apply C. Thm to $F(z) = \dfrac{f(z) - f(a)}{z - a}$ ← removable sing at a

Use $$\left(\int_{C_R} - \int_{C_r}\right)\frac{1}{z-a}\,dz = \underset{\substack{\uparrow \\ a \text{ inside } C_R}}{2\pi i} - \underset{\substack{\uparrow \\ a \text{ outside } C_r}}{0}$$

Laurent series: Take $z_0 = 0$. f analytic on $A(\rho_1, \rho_2)$

$$f(z) = \underbrace{\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{\underset{w\left(1-\frac{z}{w}\right)}{\underbrace{w-z}}}\,dw}_{=\text{ power series in } z} + \underbrace{\frac{1}{2\pi i}\left(-\int_{C_r}\frac{f(w)}{\underset{-z\left(1-\frac{w}{z}\right)}{\underbrace{w-z}}}\,dw\right)}_{I(z)}$$

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{where}$$

$$a_n = \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w^{n+1}}\, dw$$

$$I(z) = \frac{1}{2\pi i} \cdot \frac{1}{z} \cdot \int_{C_r} \underbrace{\frac{f(w)}{1 - \frac{w}{z}}}_{f(w)\left[1 + \left(\frac{w}{z}\right) + \cdots \left(\frac{w}{z}\right)^N + \underbrace{\frac{\left(\frac{w}{z}\right)^{N+1}}{1 - \frac{w}{z}}}_{E_N\left(\frac{w}{z}\right)}\right]} dw$$

$$= \sum_{n=1}^{N+1} \underbrace{\left(\frac{1}{2\pi i} \int_{C_r} f(w)\, w^{n-1}\, dw \right)}_{a_{-n} = \frac{1}{2\pi i} \int_{C_r} \frac{f(w)}{w^{-n+1}}\, dw} \frac{1}{z^n} + \mathcal{E}_N(z)$$

$\leftarrow$ same formula as a_n's of power series part!

Finally, $\displaystyle |\mathcal{E}_N(z)| \leq \frac{1}{2\pi} \frac{1}{|z|} \cdot \max_{C_r} |f| \frac{\left(\frac{r}{|z|}\right)^{N+1}}{1 - \frac{r}{|z|}} \longrightarrow 0$ as $N \to \infty$,

Get uniform convergence on sub-annuli $r < \underbrace{R_1 < R_2}_{\text{here}} < R$

Notation: $\displaystyle \sum_{n=-\infty}^{\infty} a_n z^n$ $\leftarrow$ convergence means both $\displaystyle \sum_{1}^{\infty}$ and $\displaystyle \sum_{-1}^{-\infty}$ converge.

Theorem: f analytic on $D_\rho(z_0) - \{z_0\}$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$= \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\substack{\text{unif conv on} \\ |z-z_0| < R < \rho \\ \text{analytic on } D_\rho(z_0)}} + \underbrace{\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}}_{\substack{\text{unif conv on} \\ |z-z_0| > r > 0 \\ \text{analytic on } \mathbb{C} - \{z_0\}.}}$$

where $\displaystyle a_n = \frac{1}{2\pi i} \int_{C_{\tilde{r}}} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$

$\tilde{r}$ can be anything $0 < \tilde{r} < \rho$ because of Cauchy Thm Ann.

<u>Theorem:</u> f has an isolated sing at a.

1) $a_{-n} = 0$, $n = 1, 2, 3, \ldots \iff a$ is removable.

2) Only finitely many $a_{-n} \neq 0$ (negative terms in L.E.) $\iff a$ is a pole

3) <u>Infinitely</u> many $a_{-n} \neq 0 \iff a$ is essential.

<u>Pf</u>: (1): $\Rightarrow$ f = power series. ✓ (easy).

$\Leftarrow$ $\displaystyle a_{-n} = \frac{1}{2\pi i} \int_{C_r} \underbrace{f(w)}_{\substack{\text{removable} \\ \text{at } a}} (w-a)^{n-1}\, dw = 0$ by Cauchy's Thm.

(2): Lemma: Laurent expansions are unique.

Why: $$a_N = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-a)^{N+1}}\,dz$$

$$= \frac{1}{2\pi i}\int_{C_r} \underbrace{\left[\sum_{-\infty}^{\infty} b_n (z-a)^n\right]}_{\text{unif conv on } C_r} \frac{1}{(z-a)^{N+1}}\,dz$$

$$= \sum_{-\infty}^{\infty} b_n \underbrace{\left(\frac{1}{2\pi i}\int_{C_r} (z-a)^{n-N-1}\,dz\right)}_{\text{all } = 0 \text{ except the } n=N \text{ term}}$$

$$= b_N \quad \checkmark$$

(2): $\Longrightarrow$ easy

$\Longleftarrow$ Principal part is the Laur. Exp!

(3): Left over case! ✓

Note: $$\operatorname{Res}_a f = a_{-1} = \frac{1}{2\pi i}\int_{C_r} f(w)\,dw$$

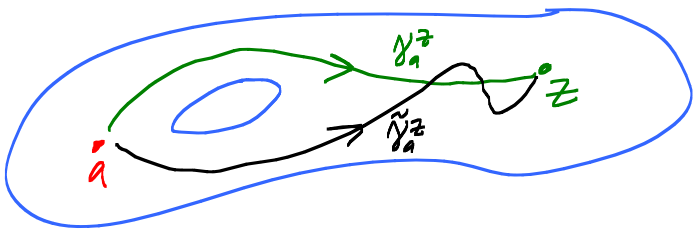
Meaning of residue: f has an analytic antiderivative on $D_\rho(a) - \{a\} \iff \operatorname{Res}_a f = 0$.

Lemma: $g = F'$ on a domain $\Omega \iff \int_\gamma g = 0$ for every closed γ in Ω.

Pf: $\Longrightarrow$ easy by Fund Thm Calc.

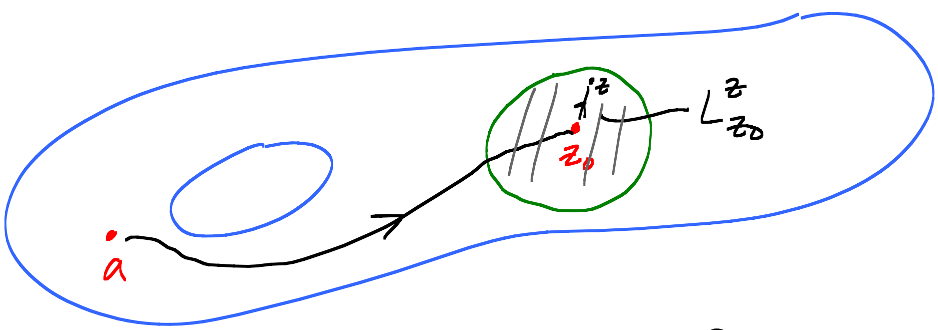
$\Longleftarrow$ Try to define $$F(z) = \int_{\gamma_a^z} g(w)\,dw$$

where γ_a^z starts at a and goes to z in Ω.



$\gamma_a^z \cup (-\tilde{\gamma}_a^z)$ is a closed curve.

See $F(z)$ is well defined.



$$\frac{d}{dz}\Big(\underbrace{\int_{\gamma_a^{z_0}} g\,dw}_{\text{const}} + \int_{L_{z_0}^z} g\,dw\Big) = 0 + g(z)$$

Pf of Thm: $f = F'$. Then $a_{-1} = \frac{1}{2\pi i}\int_{C_r} \underset{=F'}{f}\,dz = 0$ ✓

$\Longleftarrow$ Assume $\mathrm{Res}_a f = 0$.

γ closed curve in $D_\rho(a) - \{a\}$. $$\int_\gamma f\,dz = \int_\gamma \sum_{-\infty}^{\infty} a_n (z-a)^n\,dz$$

$$= \underbrace{\sum_{-\infty}^{\infty} a_n \int_\gamma (z-a)^n \, dz}_{\text{all } =0 \text{ except } n=-1. \text{ But } a_{-1}=0 \text{ kills that term}} = 0 \checkmark$$