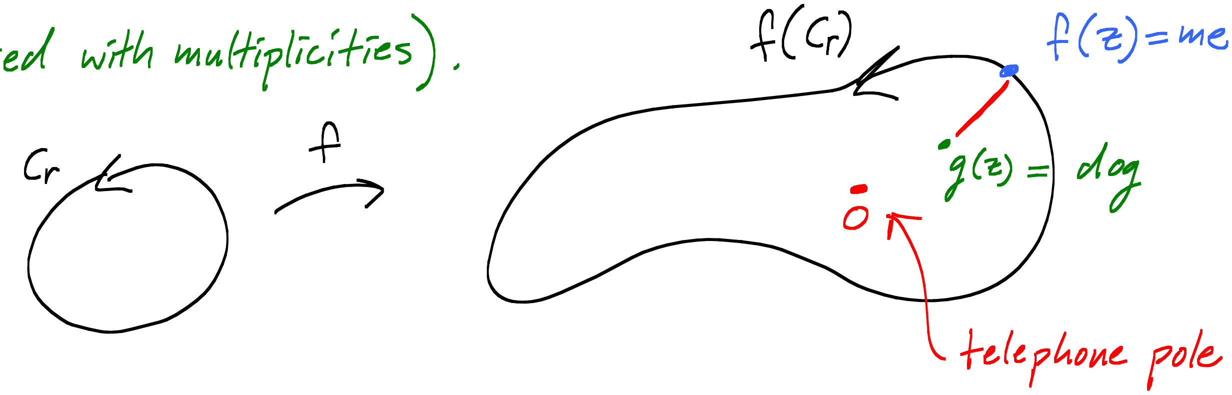
Lecture 26 Rouché's Theorem, Hurwicz Thms, Homotopy

Rouché's Theorem for a disc (or a "toy region"): Suppose f and g are analytic on $D_R(a)$ and $0 < r < R$. If $|f(z) - g(z)| < |f(z)|$ on $C_r(a)$, then f and g have the same # zeroes inside $C_r(a)$ (counted with multiplicities).

Meaning:

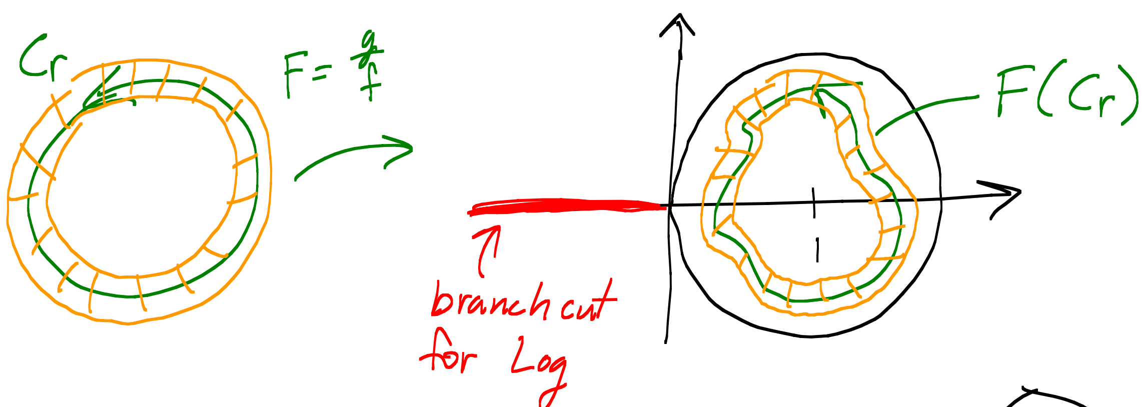


$$\underbrace{|f(z) - g(z)|}_{\text{leash length}} < \underbrace{|f(z)|}_{\text{distance from me to pole}}$$

Arg. # zeroes of f in C_r = # times $f(z)$ goes around origin
= # times dog goes around pole.

Note: Ineq $\Rightarrow$ f non-vanishing on C_R
$\Rightarrow$ g same.

Pf: Divide Ineq by $|f|$: $\left| 1 - \frac{g(z)}{f(z)} \right| < 1$



$$\int_{C_r} \frac{F'}{F}\, dz = \int_{C_r} \frac{d}{dz}(\operatorname{Log} F)\, dz = 0$$

But $$\frac{F'}{F} = \frac{g'}{g} - \frac{f'}{f}$$ ← "logarithmic diff'tion"

$$\underbrace{\frac{1}{2\pi i}\int_{C_r}\frac{F'}{F}\,dz}_{=0} = \underbrace{\frac{1}{2\pi i}\int_{C_r}\frac{g'}{g}\,dz}_{\#\text{ zeroes } g} - \underbrace{\frac{1}{2\pi i}\int_{C_r}\frac{f'}{f}\,dz}_{\#\text{ zeroes } f \text{ inside } C_r}$$

Remark: Can improve Rouché's to allow zeroes & poles.

Get [# zeroes − # poles] of f = same # for g.

Theorem (Hurwicz): Suppose f_n is a seq of analytic fcns on a domain Ω. $f_n \twoheadrightarrow f$. ← consequently f is analytic too

If all f_n's are non-vanishing on Ω, then

1) $f \equiv 0$, or

2) f non-vanishing too!

Way false $\mathbb{R} \to \mathbb{R}$:

Pf: Suppose $f(z_0) = 0$. If $f \not\equiv 0$, then z_0 is an isolated zero. So $\exists\ \overline{D_R(z_0)} \subset \Omega$ such that z_0 is the only zero of f in $\overline{D_R(z_0)}$.

Let $m = \min\{|f(z)| : z \in C_R(z_0)\}$.

$C_R(z_0)$ is compact and $f_n \to f$ unif on compacts.

So, $\exists\ N$ such that

$$|f_n(z) - f(z)| < \underset{\underset{\varepsilon}{\uparrow}}{m} \le |f(z)| \quad \text{on } C_R(z_0)$$

if $n > N$. Rouché's $\Longrightarrow$

$$\underbrace{\begin{pmatrix} \#\text{zeroes of } f_n \\ \text{inside } C_R(z_0) \end{pmatrix}}_{=0} = \underbrace{\begin{pmatrix} \#\text{ zeroes } f \\ \text{in } C_R(z_0) \end{pmatrix}}_{\ge 1}$$

Conclude, if $f \not\equiv 0$, it has no zeroes. ✓

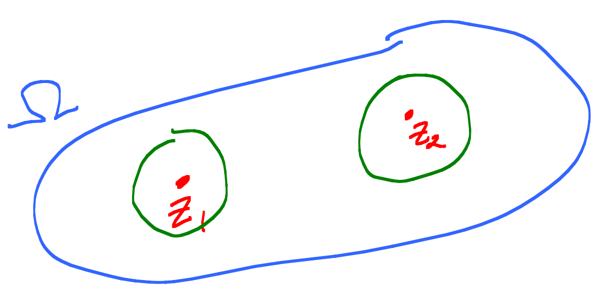
Corollary: (Hurwicz #2). $f_n \twoheadrightarrow f$.

If each f_n is one-to-one on Ω, then either

1) $f \equiv$ const, or

2) f is one-to-one too!

Pf: Redo Hurwicz #1



Suppose $f(z_1) = f(z_2) = w_0$

$f \not\equiv w_0 \Longrightarrow$

$f(z) - w_0$ has isolated zeroes at z_1, z_2

Repeat argument: $f_n - w_0$ and $f - w_0$ have same # zeroes in little discs about z_1, z_2.

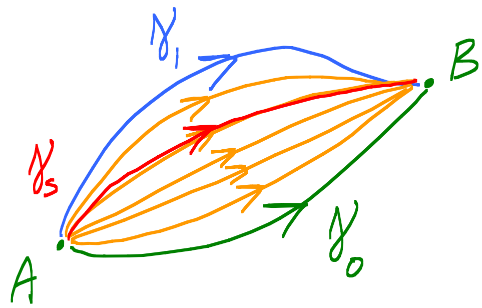
$f - w_0$ has at least one zero at z_j.

$\Rightarrow$ $f_n - w_0$ does too for $n >$ some N.

Get ζ_1, ζ_2 where $f_n(\zeta_j) = w_0$. $\Leftarrow$ not 1-1. $\lightning$

EX: $f_n(z) = z^n \longrightarrow 0$ unif on compacts of $D_1(0)$.

Homotopy:



$\gamma_j : z_j(t), \quad 0 \le t \le 1$

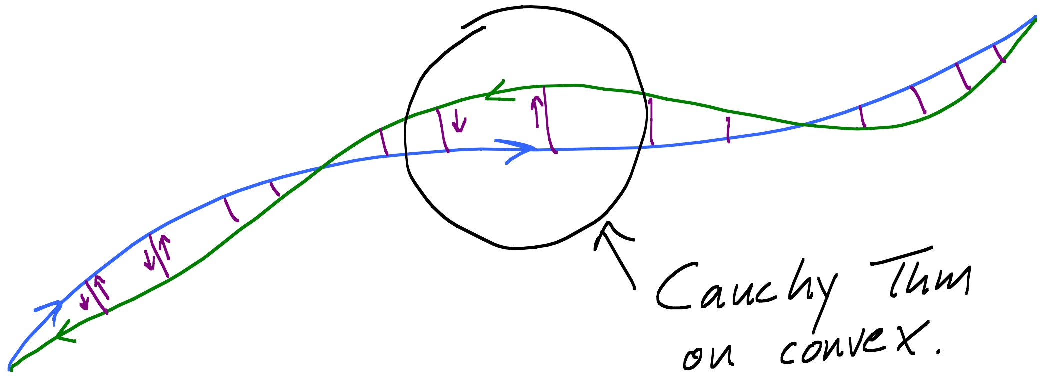
$\gamma_0 \sim \gamma_1$: γ_0 homotopic to γ_1 means there is a continuous $H : [0,1] \times [0,1] \longrightarrow \Omega$ such that

$$\begin{cases} H(0,t) = z_0(t), & 0 \le t \le 1 \\ H(1,t) = z_1(t), & 0 \le t \le 1 \end{cases}$$

$$\begin{cases} H(s,0) = A \\ H(s,1) = B \end{cases}$$

Think: $\gamma_s : z_s(t) = H(s,t), \quad 0 \le t \le 1$

Idea:

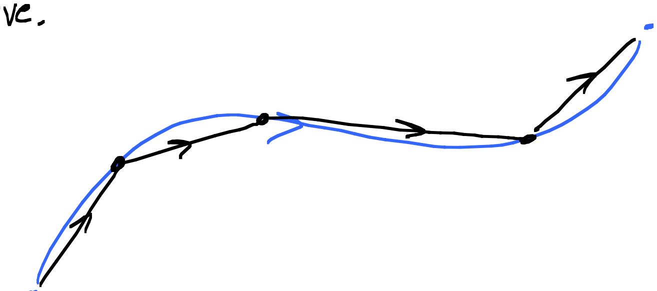


Add up. Cuts cancel. See $\int_{Blue} = \int_{Green}$.

Theorem: f analytic on Ω and $\gamma_0 \sim \gamma_1$ in Ω,

then $$\int_{\gamma_0} f\, dz = \int_{\gamma_1} f\, dz.$$

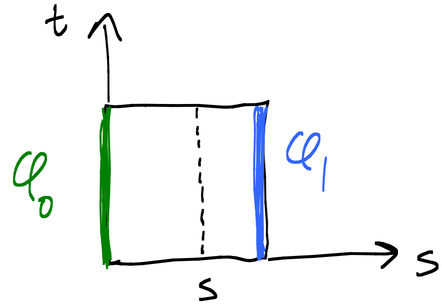
Easy fact: Can always approximate a continuous curve by a piecewise linear curve.



Remark: Can use this idea to define $\int_\gamma f\, dz$ = constant value of piecewise linear $\int$ when they get within proper ε dist to γ.

merely cont

<u>Big idea</u>:



$$\varphi_s(t) = s\,\varphi_0(t) + (1-s)\,\varphi_1(t)$$