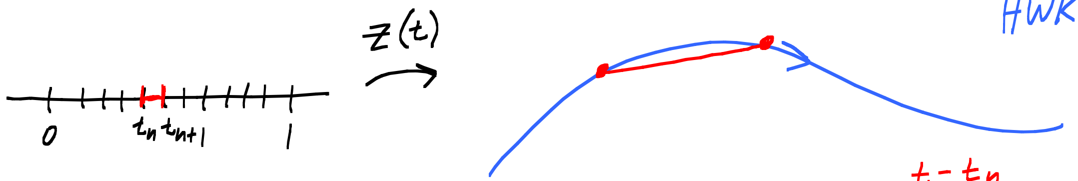
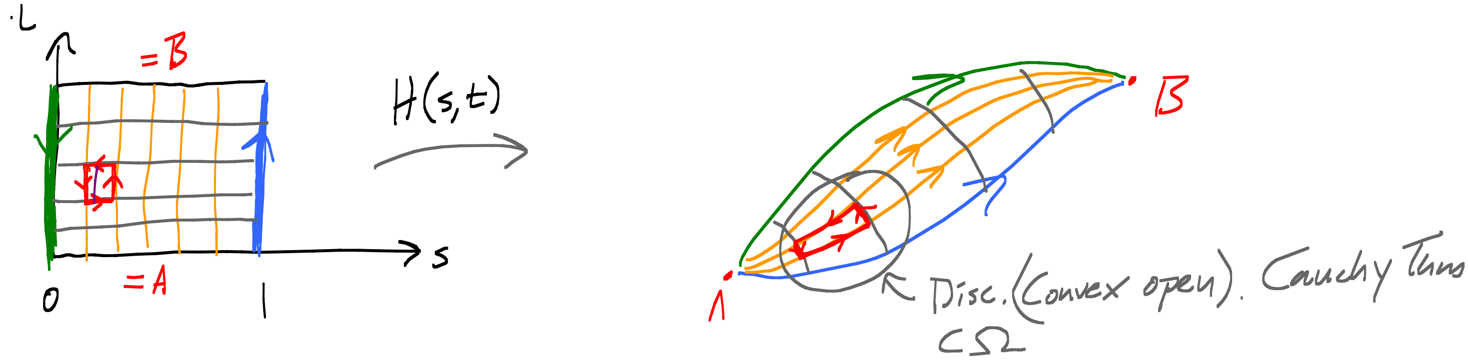
Lecture 27 Homotopy, simply connected domains, the Cauchy Theorem

HWK 7 due Mon.

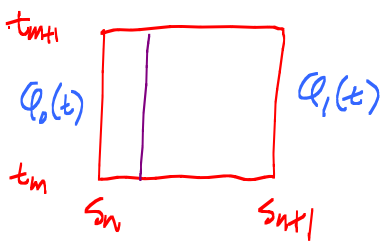


$$\tilde{z}(t) = z(t_n) + \frac{t-t_n}{t_{n+1}-t_n} \cdot \left[z(t_{n+1}) - z(t_n) \right]$$



Fact: Can approx continuous homotopies by piecewise C^1-homotopies.

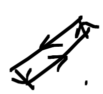
How: Set a grid. Linearize chordwise along vertical lines.



$$\varphi_0(t) + \frac{s-s_n}{s_{n+1}-s_n}\left[\varphi_1(t) - \varphi_0(t) \right]$$

Thm: f analytic on Ω. $\gamma_0 \sim \gamma_1$ in Ω. (A to B)

Then $\displaystyle\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz.$

Pf: Add up little squares as parameterizing . $\displaystyle\sum_{\text{Boxes}} \int f\,dz = 0$

Bottom $\equiv A$. Top $\equiv B$. Those $\int$'s are 0.

$$\Sigma: \left(\int_{\gamma_1} + \int_{-\gamma_0} \right) f\,dz = 0. \quad \checkmark$$

Def^n: Two closed curves γ_0 and γ_1 in Ω

are <u>homotopic</u> (in Ω) if there is a homotopy

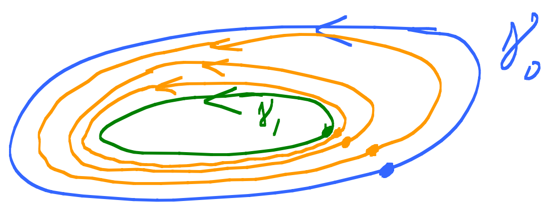
$H(s,t)$ such that $H: [0,1]\times[0,1] \to \Omega$

$$\begin{cases} H(0,t) = z_0(t) \leftarrow \gamma_0 \\ H(1,t) = z_1(t) \leftarrow \gamma_1 \end{cases}$$

Think

$\gamma_s(t) : z_s(t) = H(s,t)$

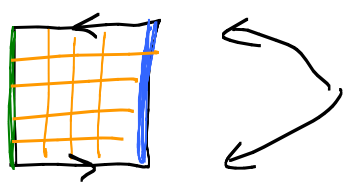
$H(s,0) = H(s,1) \leftarrow \gamma_s$ closed



<u>Big Thm</u>: f analytic, $\gamma_0 \sim \gamma_1$ (closed curves). Then

$$\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz.$$

<u>Pf</u>:



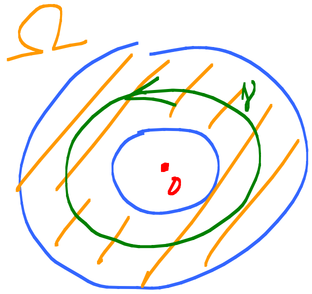
Top side cancels bottom side when Σ up boxes.

<u>Def^n</u>: A domain in $\mathbb{C}$ is <u>simply connected</u> if every closed curve in the domain is "homotopic to a pt" in the domain. $[\, z(t) \equiv z_0 ,\ 0 \le t \le 1 \,]$

<u>The Cauchy Theorem</u>: f analytic on a s.c. domain Ω. Then $\int_\gamma f\,dz = 0$ for any <u>closed</u> γ in Ω.

Pf: $\gamma \sim z_0$ $\qquad \displaystyle\int_\gamma f\, dz = \int_0^1 f(z_0)\,[\underset{\uparrow\; z'(t)}{0}]\,dt = 0$ $\qquad z(t) \equiv z_0$

EX:

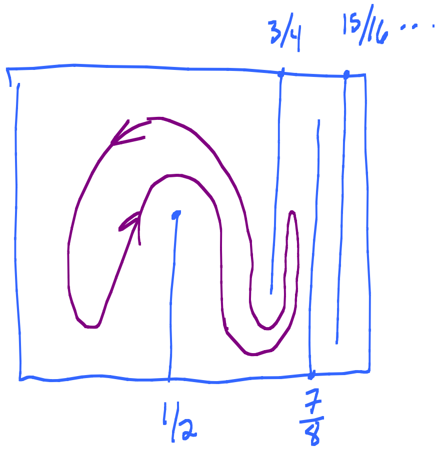


not s.c.

$$\int_\gamma \frac{1}{z}\,dz = 2\pi i \neq 0$$

Holes ruin Cauchy Thm on non-s.c. Ω

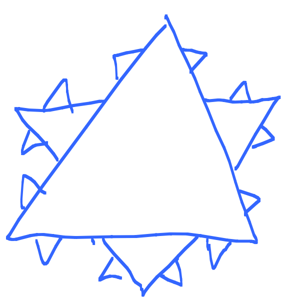
EX:



Homotopy: Suck in spaghetti.

s.c.

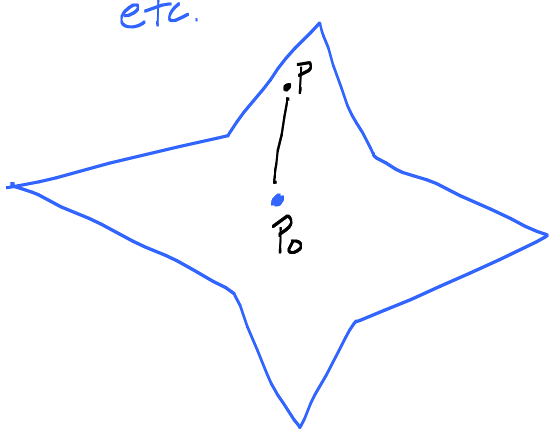
Serpienski snowflake



c.c.

$$\underset{\substack{\uparrow\\ \text{bounded!}}}{A} \leq \frac{L^2}{4\pi} \longrightarrow \infty$$

etc.



Starshaped domains are s.c.

So are <u>convex</u> domains.

Thm: Suppose f is analytic on a s.c. domain Ω.

A) f has an analytic antiderivative F on Ω.

B) If f is non-vanishing, the $\exists$ analytic G and H such that

$$f = e^G \qquad\qquad f = H^N \leftarrow H = e^{G/N}$$

G = branch of $\log f$

C) If u is harmonic on Ω, then u has a harmonic conj v on Ω.

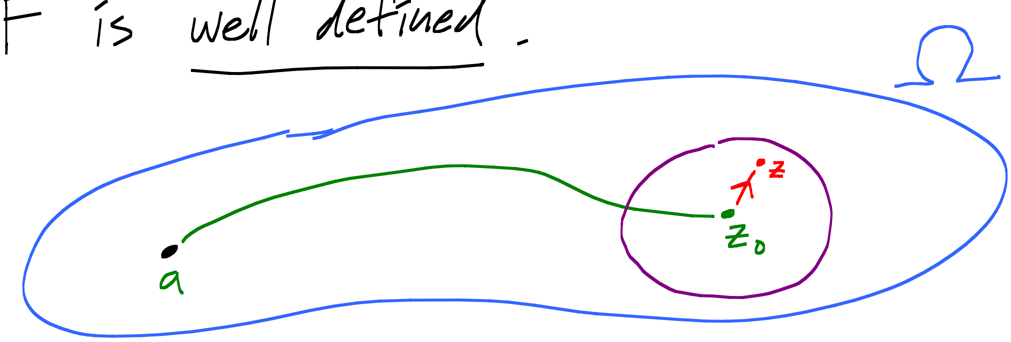
Pf: A)

$$F(z) = \int_{\gamma_a^z} f(w)\, dw$$

$\gamma_a^z \cup (-\tilde{\gamma}_a^z)$ is a <u>closed</u> curve.

<u>The</u> Cauchy thm $\Rightarrow$ $\int_{\gamma_a^z} f\, dz = \int_{\tilde{\gamma}_a^z} f\, dz$ ✓

F is <u>well defined</u>.



$$\frac{d}{dz}\left(\underbrace{\int_{\gamma_a^{z_0}} f \, dw}_{=0} + \underbrace{\int_{L_{z_0}^{z}} f \, dw}_{f(z)} \right) \checkmark$$

B) $g(z) = \int_{\gamma_a^z} \frac{f'(w)}{f(w)} \, dw$ $\boxed{g' = \frac{f'}{f}}$

f non-vanishing

$$\frac{d}{dz}\left(\frac{e^g}{f}\right) \equiv 0$$

$G = g + c$

$$f = \underbrace{K}_{e^c} e^g$$

C) u harmonic.

C-R eqns $\Rightarrow$

$F = u_x - i u_y$ is analytic.

Think $f = u + iv$

$f' = u_x + i v_x$

$= u_x - i u_y$

(A) $\Rightarrow \exists \; f$ with $f' = F$

$f = \tilde{u} + i\tilde{v}$ $\qquad \tilde{u}_x - i\tilde{u}_y = u_x - i u_y$

$\nabla \tilde{u} \equiv \nabla u \Rightarrow u = \tilde{u} + c$

Aha! $\underbrace{(\tilde{u} + c)}_{u} + i\tilde{v}$ is analytic

harm conj for u.

For next time: Review the Arzela-Ascoli Thm.:

Baby Rudin: p. 158

Ahlfors: p. 219-227

Stein: p. 225