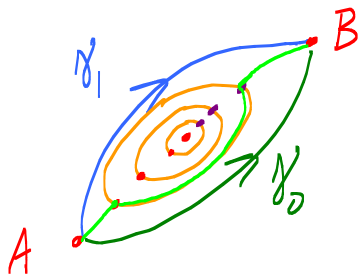
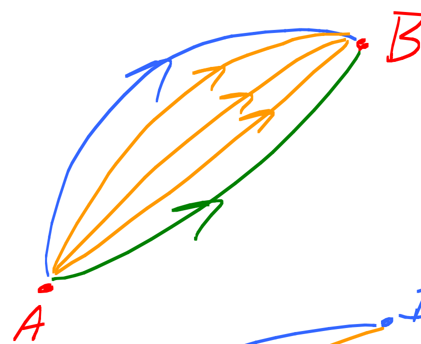
Lecture 28 Montel's Theorem

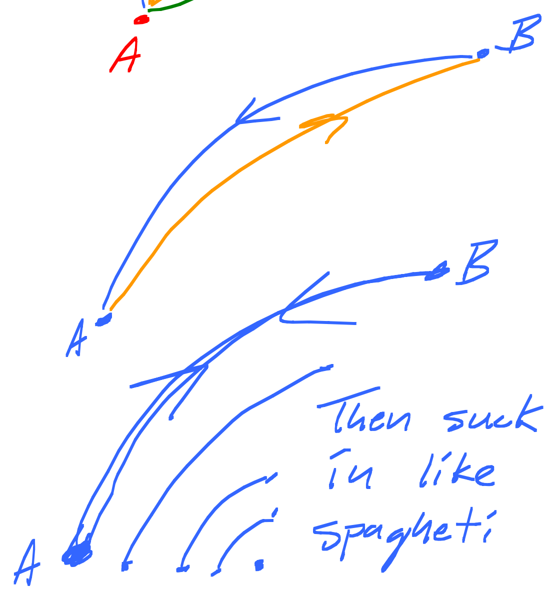
Fact: Ω is simply connected $\Longleftrightarrow$ any two curves that start and stop at $A, B \in \Omega$, resp., are homotopic.

$(\Rightarrow)$



$(\Leftarrow)$





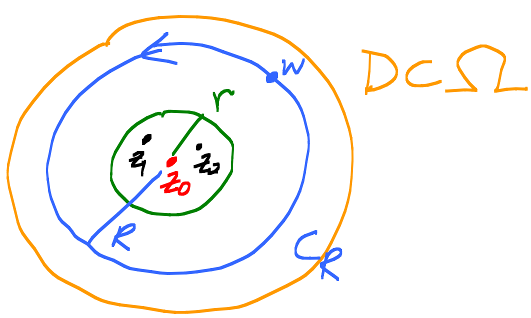
Cor: Ω a s.c. domain
f analytic on Ω.

Then $\displaystyle\int_{\gamma_A^B} f\, dz$ only

depends on A and B.

Lemma: Uniformly bounded families of analytic fcns on a domain are uniformly equicontinuous on compact subsets.

Why:



$f \in \mathcal{F}$

$$\left| f(z_1) - f(z_2) \right| = \left| \frac{1}{2\pi i} \int_{C_R} f(w) \left[\frac{1}{w - z_1} - \frac{1}{w - z_2} \right] dw \right|$$

$$= |z_1 - z_2| \frac{1}{2\pi} \left| \int_{C_R} \frac{f(w)}{(w-z_1)(w-z_2)} dw \right|$$

$$\leq |z_1 - z_2| \frac{1}{2\pi} \underbrace{\left(\max_{C_R} |f| \right)}_{\text{Bounded indep of } f \in \tilde{\mathcal{F}}!} \frac{1}{(R-r)^2} \cdot 2\pi R$$

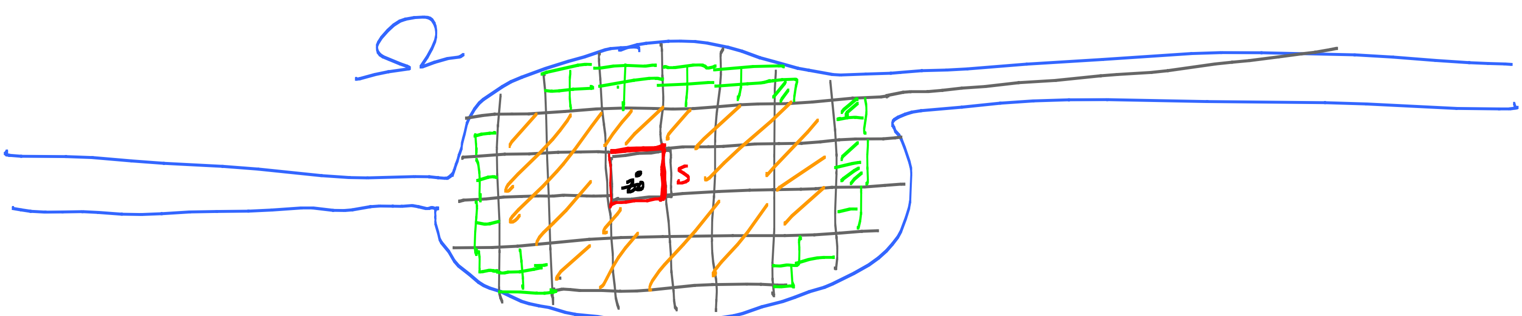
Given $\varepsilon > 0$, get a δ that works for all $f \in \tilde{\mathcal{F}}$. equi ✓

Last: Given $K \subset\subset \Omega$ (compact), cover by $D_r(z_0)$ . . .

Montel's Thm: Suppose f_n are analytic on a domain Ω and are uniformly bounded on compact sets of Ω. Then there is a subseq f_{n_k} that converges uniformly on compacts to a fcn f (which we know is analytic).

Pf: Step 1: "Exhaust" Ω by compact sets:

$$K_1 \subset K_2 \subset K_3 \subset \cdots \quad \text{and} \quad \bigcup K_j = \Omega.$$



K_1 = union of squares in $\Omega \cap D_{3s}(z_0)$

K_2 = union of squares of side $\frac{s}{2}$ $\subset\subset \Omega \cap D_{2(3s)}(z_0)$

⋮

Step 2: Fix compact K_j. Let $\{z_n\}$ be a countable dense subset of K_j: e.g. $\{z \in K_j : \operatorname{Re} z, \operatorname{Im} z \in \mathbb{Q}\}$.

Step 3: $f_1(z_1), f_2(z_1), f_3(z_1), \ldots$ is bdd seq.

So $\exists$ conv subseq:

$$f_{1,1}, f_{1,2}, f_{1,3}, \ldots \quad \text{with } f_{1,k}(z_1) \text{ converges.}$$

Next: Take subseq of $f_{1,1}(z_2), f_{1,2}(z_2), \ldots$ that conv.

get $f_{2,1}, f_{2,2}, f_{2,3}, \ldots$

Continue...

Aha! $F_n = f_{n,n}$ converges at each z_n!

Step 4: Claim: F_n are uniformly Cauchy on K_j.

$$|F_n(z) - F_m(z)| = |\underbrace{F_n(z) - F_n(z_j)}_{\substack{\text{pick } z_j \text{ close} \\ \text{to } z \text{ to make} \\ \text{small. (Using} \\ \text{equi continuity)}}} + \underbrace{F_n(z_j) - F_m(z_j)}_{\substack{F_n(z_j) \text{ converges!} \\ \text{so seq is} \\ \text{Cauchy.}}} + \underbrace{F_m(z_j) - F_m(z)}_{\substack{\text{works here} \\ \text{too!}}}|$$

Hmmm. Can get a good estimate on disc $z \in D_\delta(w_k)$. Cover K_j by these and take a finite subcover. Get a uniform uniformly Cauch estimate on K_j.

[Uniformly Cauchy $\Rightarrow$ uniform convergence]

Step 5: $F_{1,1}, F_{1,2}, \ldots$ conv unif on K_1

subseq $F_{2,1}, F_{2,2}, \ldots$ conv unif on K_2

subseq $F_{3,1}, F_{3,2}, \ldots$ conv unif on K_3

$\vdots$

$F_{n,n}$ converges unif on all K_j's.

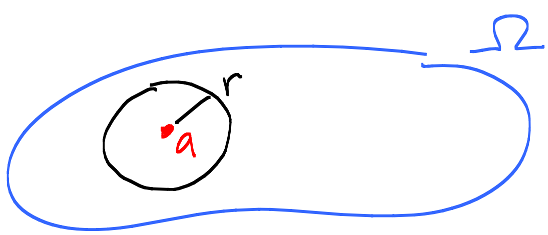
Step 6: Finally, given compact K, there is a K_j that contains.

Riemann Mapping Thm plan: $\Omega \neq \mathbb{C}$ s.c. domain.

Pick $a \in \Omega$. [Assume Ω is bnded today].

$\mathcal{F} = \{ h : \Omega \to D_1(0) \text{ analytic and } 1\text{-}1 \}$.

Note: $\mathcal{F} \neq \emptyset$ because $\frac{z}{R} \in \mathcal{F}$. ($R$ ← big)



Cauchy estimate

$$|h'(a)| \leq \frac{1}{r}$$

1 ← $\max_{\Omega} |h|$

unif bnd $h \in \mathcal{F}$.

So $M = \sup\{ |h'(a)| : h \in \mathcal{F} \}$ is finite.

Take seq $h_n \in \mathcal{F}$ so that $|h_n'(a)| \to M$.

Montel's: $\exists$ Subseq $h_{n_k} \xrightarrow[\text{unif on compacts}]{} f$ ← The Riemann map!

$|f'(a)| = M \neq 0$. So f <u>not</u> const.

Hurwicz $\Rightarrow$ f <u>1-1</u>.

Last step: Show f is <u>onto</u>.

Need Schwarz Lemma and analytic square roots of non-vanishing fcns on Ω.