Lecture 29 Riemann Mapping Theorem

HWK 7 due Mon

Thm: Ω a simply connected domain $\Omega \neq \mathbb{C}$. There is an analytic (conformal) map $f: \Omega \to D_1(0)$ that is one-to-one and onto.

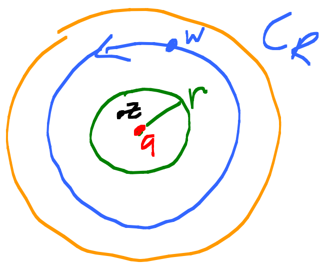
Lemma: $f_n \rightrightarrows f$ on Ω. Then $f_n^{(k)} \rightrightarrows f^{(k)}$.

analytic

Way false $\mathbb{R} \to \mathbb{R}$. $h_n(t) = \frac{1}{2^n} \sin(2^n t) \rightrightarrows$ Nowhere diff'ble fcn!

C^∞ smooth

Pf:



$$\left| f_n^{(k)}(z) - f^{(k)}(z) \right| = \left| \frac{k!}{2\pi i} \int_{C_R} \frac{f_n(w) - f(w)}{(w-z)^{k+1}} \, dw \right|$$

$$\leq \frac{k!}{2\pi} \frac{\max_{C_R} |f_n - f|}{(R-r)^{k+1}} \, 2\pi R$$

$\longrightarrow 0$ as $n \to \infty$

(C_R is compact).

Given compact $K \subset\subset \Omega$. Cover K by green $D_r(a)$. Take finite subcover. ✓

Pf of RMT: Let $\tilde{\mathcal{F}}$ = family of analytic $f: \Omega \to D_1(0)$ that are one-to-one.

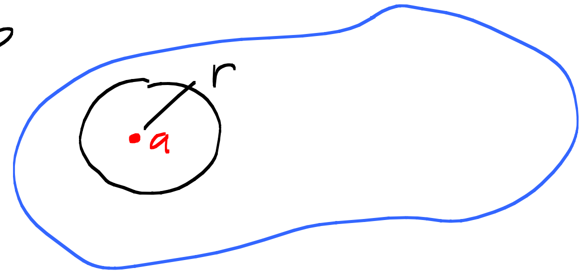
Claim: $\tilde{\mathcal{F}}$ not empty. [Pf later. Easy if Ω is bdd. Then $f = \frac{z}{R}$ is in in $\tilde{\mathcal{F}}$ if $\Omega \subset D_R(0)$.]

Pick $a \in \Omega$. Let $M = \sup \{ |f'(a)| : f \in \tilde{\mathcal{F}} \}$.

Step 1: $0 < M < \infty$

$\uparrow$ f 1-1

$\Longrightarrow f'$ non-vanishing



Cauchy estimate: $\displaystyle |f'(a)| \le \frac{\overset{!!}{\max_{C_r(a)}} |f|}{r} \le \frac{1}{r}$

[Can let $r \nearrow \operatorname{dist}(a, b\Omega)$ to see $\displaystyle |f'(a)| \le \frac{1}{\operatorname{dist}(a, b\Omega)}$]

Step 2: Take a seq $f_n \in \mathcal{F}$ such

$$|f_n'(a)| \longrightarrow M \quad \text{as} \quad n \to \infty.$$

Montel's Thm $\Longrightarrow$ $\exists$ subseq $f_{n_k} \rightrightarrows F$ on Ω.

[F is the Riemann map!]

Step 3: Today's lemma $\Longrightarrow$ $f_{n_k}'(a) \to F'(a)$.

So $|F'(a)| = M \leftarrow$ not zero.

So $F \not\equiv$ const.

Hurwicz's Thm $\Longrightarrow$ F is one-to-one.

Step 4. $F \in \mathcal{F}$. Note: $f_n : \Omega \to D_1(0)$

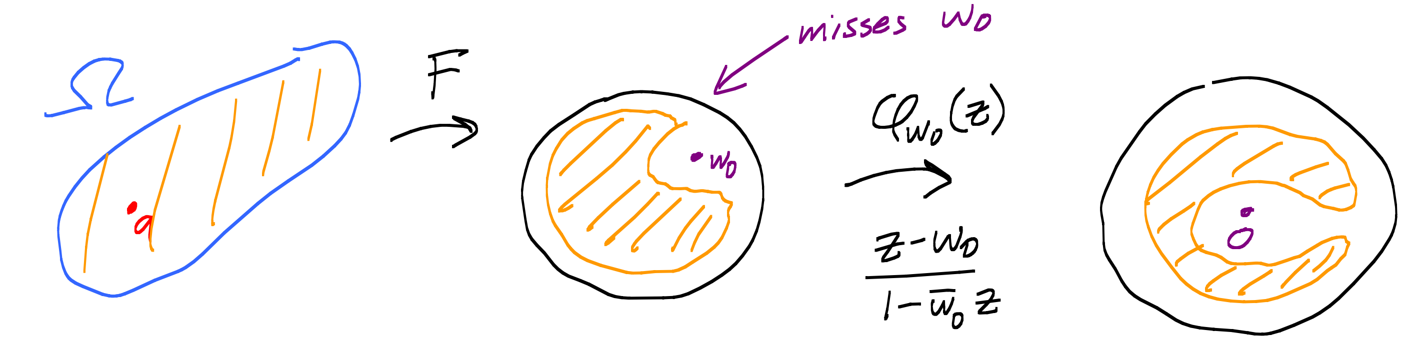
$|f_n(z)| < 1$

$\Longrightarrow |F(z)| \le 1$

But if $|F(z)|=1$, Max princ $\Rightarrow F \equiv$ const. ↯.

So $|F(z)|<1$. $F : \Omega \to D_1(0)$, 1-1, analytic. ✓

Step 5: F is onto. Suppose not.



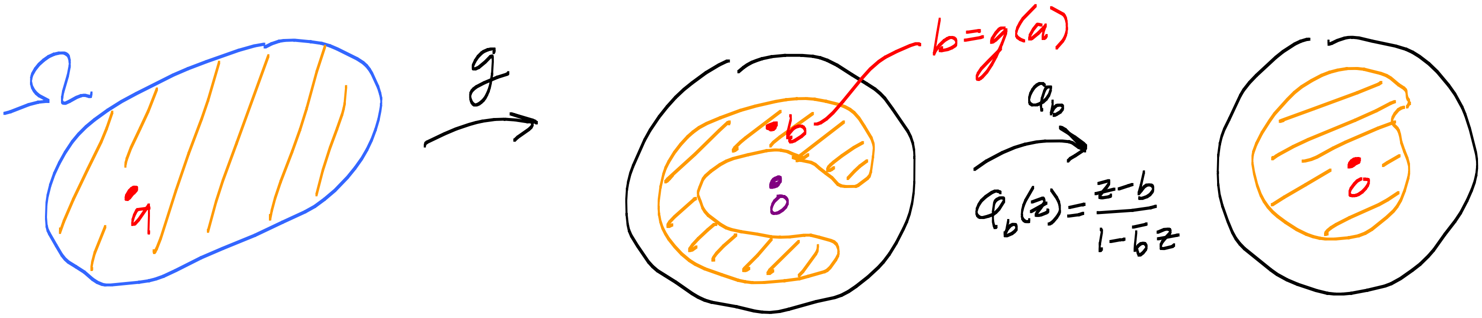
Note: $\varphi_{w_0}(F(z))$ is non-vanishing analytic on Ω.

So $\exists$ analytic g on Ω with $g^2 = \varphi_{w_0} \circ F$.

Claim: $g \in \mathcal{F}$. $g : \Omega \to D_1(0)$ easy. $\left(|g^2|<1 \Rightarrow |g|<1 \right)$

g^2 1-1 $\Rightarrow g$ is 1-1.

Why: $g(z_1)=g(z_2) \Rightarrow g^2(z_1)=g^2(z_2) \Rightarrow z_1=z_2$.



Claim: $\varphi_b \circ g \in \mathcal{F}$ ✓ and the derivative at a is strictly bigger than $M = \text{Max}$. $\lightning$. No such w_0.

Write $\tilde{F} = \varphi_b \circ g$

Check that $F = G \circ \tilde{F}$

$$G = \varphi_{w_0}^{-1} \circ s \circ \varphi_b^{-1}$$

where $s(z) = z^2$

Note: $G : D_1(0) \to D_1(0)$ analytic.

Cor to Schwarz: $|G'(0)| \le 1 \leftarrow$ if $=1$, $G(z) = \lambda z$.

G not 1-1 (because of $s(z) = z^2$) $\Longrightarrow$

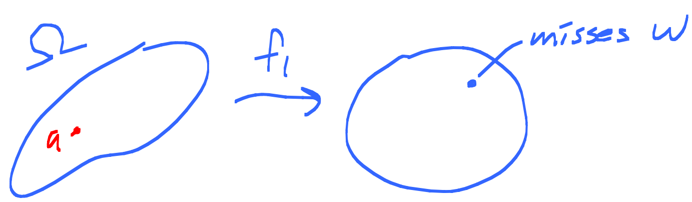
$|G'(0)| < 1$.

Finally $$F'(a) = G'(\underbrace{\tilde{F}(a)}_{=0})\, \tilde{F}'(a)$$

$$\underbrace{|F'(a)|}_{M} = \underbrace{|G'(0)|}_{<1}\, \underbrace{|\tilde{F}'(a)|}_{> M\,!} \quad \lightning.$$

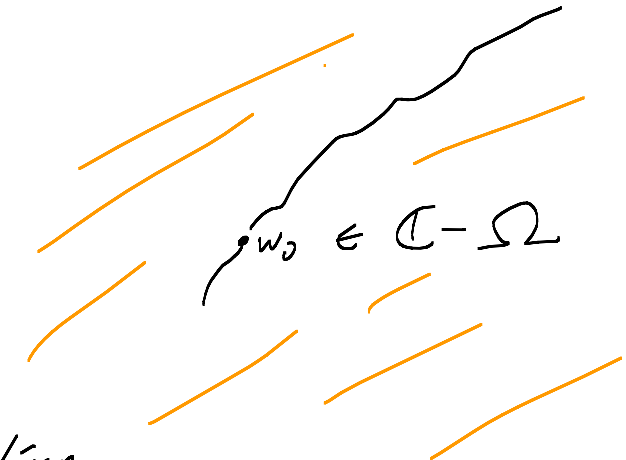
So F is onto.

Koebe:



Do the routine.
Get f_2 with $|f_2'(a)|$ bigger. Repeat.

Get seq $\longrightarrow$ Riemann map.



Last piece: $\tilde{\mathcal{F}} \neq \emptyset$.

$\Omega \neq \mathbb{C}$. $w_0 \in \mathbb{C} - \Omega$

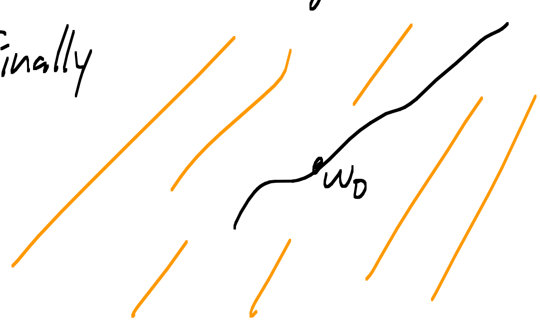
$z - w_0$ is analytic non-vanishing on Ω. So $\exists$ analytic g with $g(z)^2 = \underbrace{z - z_0}_{1-1}$

So g is 1-1.

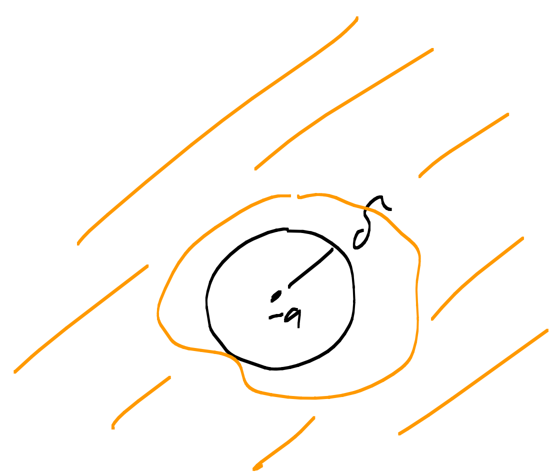
Claim: g misses a disc.

why: Open Mapping Thm $\Rightarrow$ $g(\Omega)$ contains a disc $D_\delta(a)$. Now $-D_\delta(a)$ is not in $g(\Omega)$ because g^2 is 1-1.

Finally



$\xrightarrow[1-1]{g}$



$\xrightarrow[1-1]{\frac{\delta}{z+a}}$

Get composition in $\tilde{\mathcal{F}}$ using Casorati-Weierstraß Thm idea.