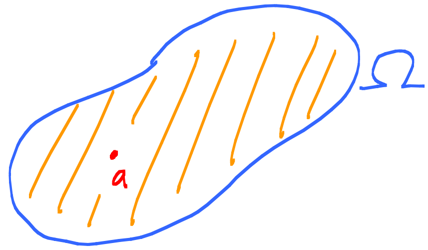
Lecture 30 Harmonic fcns



$f_a : \Omega \to D_1(0)$ Riemann map

$|f_a'(a)| = M = \sup\{|h'(a)| : h : \Omega \to D_1(0),\ 1\text{-}1, \text{ analytic}\}$

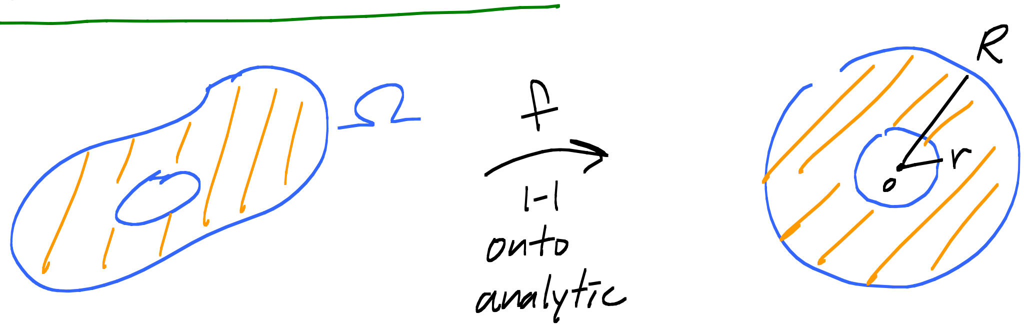
Remarks: It must be that $f_a(a) = 0$.

Why: If $f_a(a) = b$, then consider $\varphi_b \circ f$ where $\varphi_b(z) = \frac{z-b}{1-\bar{b}z}$.

Now $\varphi_b'(\underbrace{f_a(a)}_{b})\, f_a'(a) = \underbrace{\frac{1}{1-|b|^2}}_{>1} \cdot f_a'(a)$ $\lightning$ So $b = 0$.

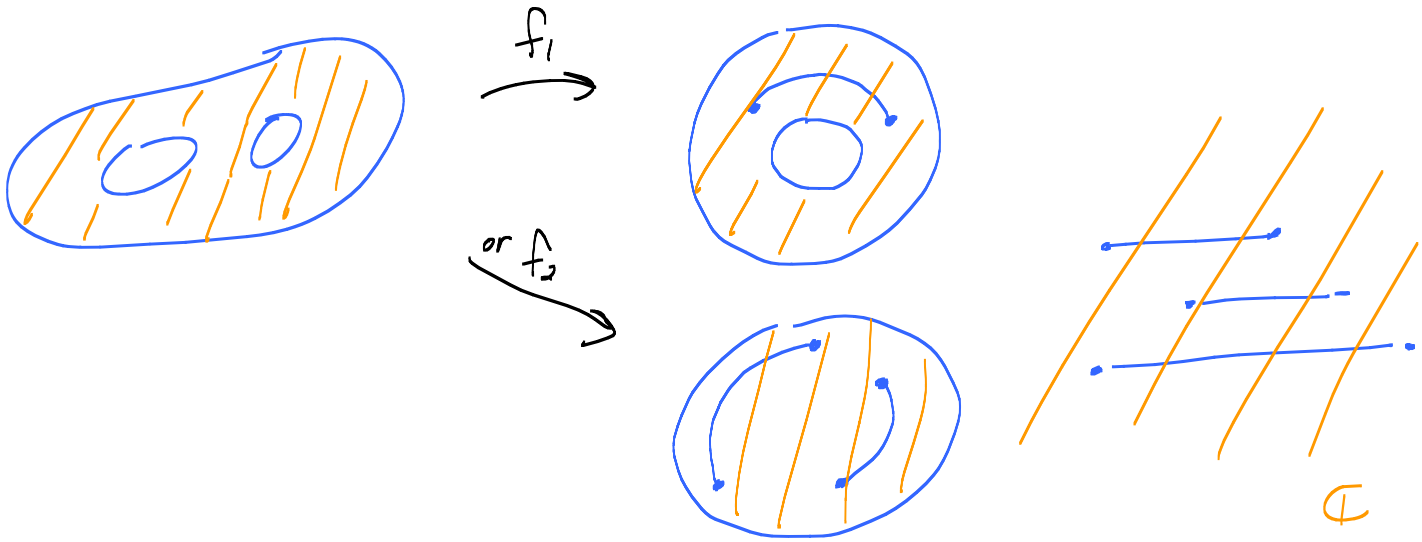
Fact: Specifying $f_a'(a) \in \mathbb{R}^+$ makes Riemann map uniquely determined.

What about domains with holes?

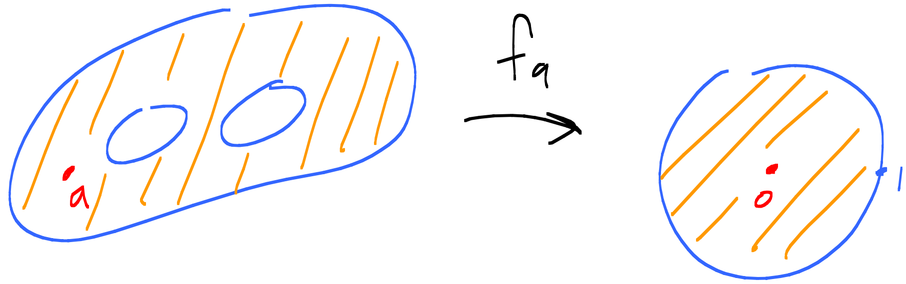


$\frac{r}{R}$ determines the mapping class.

($\frac{r}{R}$ = the "modulus"



Or



$\mathcal{F}$: $\{h : \Omega \to D_1(0),$ analytic$\}$.
(not assuming 1-1)

Get f_a maximizing $|h'(a)|$.

Ahlfors map.

Ω n-connected, Ahlfors map is onto, but not 1-1.
It is n-to-one, counting multiplicities.

Harmonic functions: u is harmonic means u real valued, $\mathcal{C}^2$-smooth, and $\Delta u \equiv 0$ on a domain.

Know: 1) Restrict to a disc. Can cook up a harmonic conjugate v so that $\underbrace{u+iv}_{f}$ is analytic.

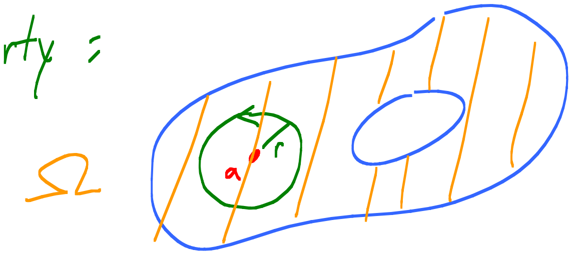
Know f is infinitely complex diff'ble.

So u, v are $\mathcal{C}^\infty$-smooth.

Equivalent def of harmonic: u is locally the real part of an analytic fcn.

Remark: Can get global harmonic conj on a simply conn. domain, but holes lead to problems. [$\ln|z|$ on $\mathbb{C} - \{0\}$]

Important fact: Harmonic functions satisfy the averaging property:



$$u(a) = \underbrace{\frac{1}{2\pi r}}_{\text{length } C_r} \int_0^{2\pi} u(a+re^{i\theta})\, \underbrace{r\,d\theta}_{ds}$$

Why: $f = u + iv$. Cauchy integral formula:

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{\underbrace{(a+re^{i\theta})}_{z(\theta)} - a} \underbrace{ire^{i\theta}\, d\theta}_{z'(\theta)\, d\theta}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{i\theta})\, d\theta$$

Take real part.

Big fact: Continuous real valued fcns that satisfy the ave property must be harmonic!

Thm: Ave prop $\Rightarrow$ Max Princ.

Dirichlet Problem on $D_1(0)$: Given a real valued continuous fcn $\varphi(\theta)$ on unit circle $[\varphi(0) = \varphi(2\pi)]$, find a real valued fcn u that is

1) continuous on $\overline{D_1(0)}$,

2) harmonic on $D_1(0)$, and

3) $u(e^{i\theta}) = \varphi(\theta)$.

<u>Solution:</u> Given by the Poisson kernel

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|a|^2}{|a-e^{i\theta}|^2}\,\varphi(\theta)\,d\theta$$

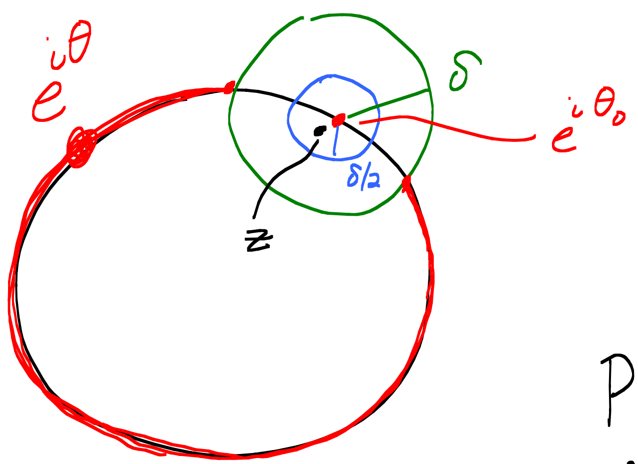
Poisson kernel: $P(a,\theta) = \frac{1}{2\pi}\,\frac{1-|a|^2}{|a-e^{i\theta}|^2}$

<u>Three big properties of $P(r,\theta)$:</u>

1) $P(a,\theta) > 0$ $\quad a \in D_1(0),\ 0 \le \theta \le 2\pi$

2) $\int_0^{2\pi} P(a,\theta)\,d\theta = 1$ for any $a \in D_1(0)$

3)



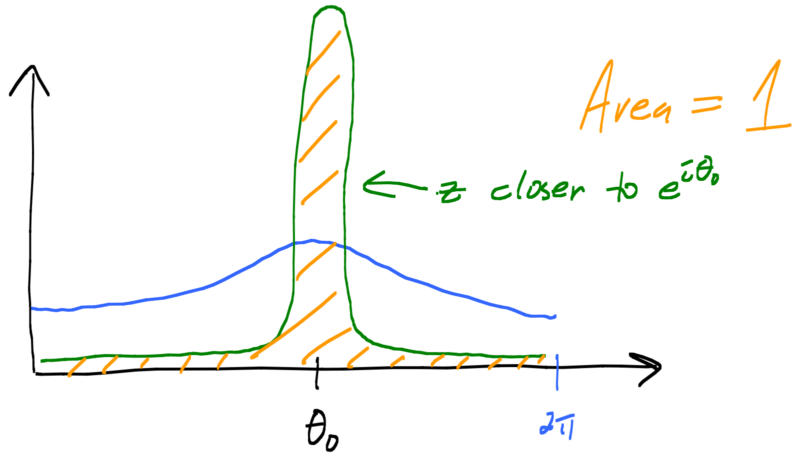
$I_\delta = \{\theta : e^{i\theta} \notin D_\delta(e^{i\theta_0})\}$

$P(z,\theta) \to 0$

<u>uniformly</u> on I_δ

as $z \to e^{i\theta_0}$ in $D_{\delta/2}(e^{i\theta_0})$.

In fact $P(z,\theta) \le \frac{4}{\pi\delta^2}(1-|z|)$



4) $P(a, \theta)$ is harmonic in a.

Hmmm: Say u solves prob. Ave prop $\Rightarrow$

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta \xrightarrow[r\to 1]{} \frac{1}{2\pi}\int_0^{2\pi} \varphi(\theta)\,d\theta$$

Aha! Can use a Möbius trans to move a to the origin.

$$\varphi_{-a}(z) = \frac{z+a}{1+\bar{a}z} \qquad u(a) = u(\varphi(0))$$