Lesson 32 : Harmonic functions

HWK 8: Prob 1. Formula defines the Bernoulli #s.

Only need to show : 1) even power coeff all vanish in Σ

2) Compute B_1, B_2

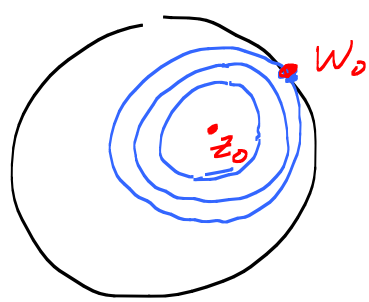
Ave Property $\Longrightarrow$ Max Princ.

Suppose u is cont on $\overline{D_1(0)}$ and satisfies ave prop inside.

Then $\displaystyle \max_{\overline{D_1(0)}} u = \max_{C_1(0)} u$

Case 1 : $u(z_0) = M \leftarrow$ max on $\overline{D_1(0)}$. $z_0 \in C_1(0)$. ✓

Case 2 : $z_0 \in D_1(0)$.



$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta \qquad \text{Let } r \nearrow \underbrace{1-|z_0|}_{R}.$$

$$\| \qquad\qquad \downarrow$$

$$M = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{u(z_0 + Re^{i\theta})}_{\leq M}\,d\theta \leftarrow \text{must be that } |u(z_0 + Re^{i\theta})| \equiv M$$

Conclude $u(w_0) = M$ too!

Hmmm. Same reasoning shows that $u \equiv M$ on $D_R(z_0)$.

$\Longrightarrow \quad u \equiv M$ on $D_1(0)$.

Similarly: Just a local max inside $\Omega \Rightarrow \equiv$ const on Ω.

Note: Max princ for $-u \Rightarrow$ Min Princ for u.

Remark: Max Princ for analytic $\Rightarrow$ Max Princ for harmonic.

u harmonic, local max at $z_0 \in \Omega$. Take $\overline{D_\varepsilon(z_0)} \subset \Omega$.

Get $f = u + iv$ on $D_\varepsilon(z_0)$ analytic.

$$e^u = e^u |e^{iv}| = \underbrace{|e^{u+iv}|}_{\text{local max at } z_0} \quad \Rightarrow \quad e^f \equiv c \text{ on } D_\varepsilon(z_0).$$

But $u = \operatorname{Ln} e^u = \operatorname{Ln}|f| \equiv \operatorname{Ln}|c|$ on $D_\varepsilon(z_0)$

$\Rightarrow u \equiv$ const on Ω,

Ave Prop $\Rightarrow$ Harmonic: Suppose u is a continuous real valued fcn that satisfies the ave prop on Ω. Then u is harmonic.

Pf: Can restrict attn to a disc. Can rescale and trans to assume u harmonic on $D_R(0)$, $R > 1$.

Big idea: Let $\tilde{u}$ = Poisson integral of $u(e^{i\theta})$.

Let $v = u - \tilde{u}$ ← satisfies ave prop! So Max princ too.

$\equiv 0$ on $C_1(0)$

Max princ $\Rightarrow$ $u-\tilde{u} \le 0$ on $\overline{D_1(0)}$.

[Ave prop for u $\Rightarrow$ Ave prop for $-u$.]

Repeat using $\tilde{u}-u$. Get $\tilde{u}-u \le 0$. So $u \equiv \tilde{u}$

harmonic!

Theorem: Suppose is u is

1) continuous

2) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ exist, and

3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0$ on Ω.

Then u is harmonic (locally $=$ Re $f \leftarrow$ analytic)

Pf: Can assume u satisfies hyp on $D_R(0)$, $R>1$.

Let $\tilde{u}(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2}\, u(e^{i\theta})\, d\theta.$

Want to see $\tilde{u} \equiv u$.

If not, $\exists$ $z_0 \in D_1(0)$ where $\tilde{u}(z_0) - u(z_0) \ne 0$.

Suppose >0. [Note: $\tilde{u}-u \equiv 0$ on $C_1(0)$.]

Say $\tilde{u}(z_0) - u(z_0) = m$

Sneaky trick: $\varepsilon|z|^2 = \varepsilon(x^2+y^2)$ has a stricty >0 Δ.

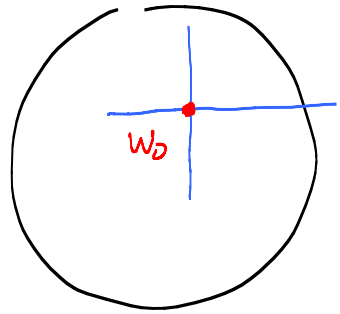
$\Delta(\varepsilon|z|^2) = 4\varepsilon > 0$

Let $v = \tilde{u} - u + \frac{m}{2}|z|^2 \leftarrow \Delta > 0$

$= \frac{m}{2}$ on bndry.

$$v(z_0) = m + \frac{m}{2}|z_0|^2 > m.$$

Conclude, $\exists \ w_0 \in D_1(0)$ where v has its absolute max on $\overline{D_1(0)}$.



2nd derivative test in horiz and vert dir.

$$\left[h''(t_0) > 0 \implies \text{local strict min.} \right]$$

Aha! Since $v(w_0)$ is a max, 2nd der test

$\implies \frac{\partial^2}{\partial x^2} v \le 0$ at w_0. $\leftarrow$ horiz

$\frac{\partial^2}{\partial y^2} v \le 0$ $\leftarrow$ vert

$+ \quad \Delta v\big|_{w_0} \le 0$ ↯ because $\Delta v > 0$.

No such place where $\tilde{u} - u$ is > 0.

Similary, repeat using $u - \tilde{u}$ if its < 0 at a pt.

Conclude $u \equiv \tilde{u}$. ✓

Riemann removable singularity theorem for harmonic fcns:

Suppose u is harmonic on $D_\varepsilon(a) - \{a\}$ and $|u|$ bdd there.

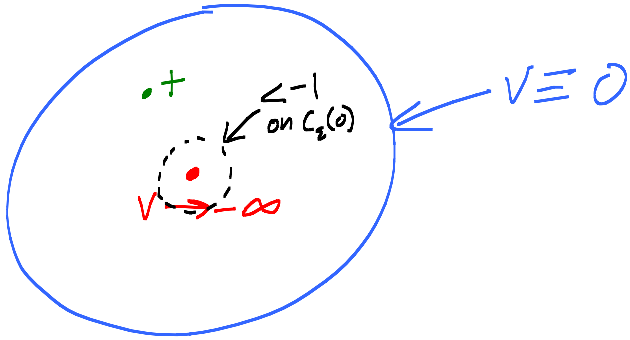
Then u can be a value at a to make it harmonic on $D_\varepsilon(a)$.

Pf: Can assume u harmonic on $D_R(0) - \{0\}$, $R > 1$.

Let $\tilde{u}$ = Poisson int of bndry values of u on $C_1(0)$.

Know $\tilde{u} \equiv u$ on $C_1(0)$, u is harmonic on $D_1(0)$ and continuous on $\overline{D_1(0)}$.

Sneaky trick: Define $v = u - \tilde{u} + \varepsilon \operatorname{Ln}|z|$



Max princ $\Longrightarrow$ $v \leq 0$ on $\overline{D_1(0)} - \{0\}$.

Whoa! True for any $\varepsilon > 0$! $\quad u - \tilde{u} + \varepsilon \operatorname{Ln}|z| \leq 0$

Fix z. Let $\varepsilon \to 0$. Get $u - \tilde{u} \leq 0$.

Repeat using $\tilde{u} - u$ in place of $u - \tilde{u}$. See $u \equiv \tilde{u}$ on $\overline{D_1(0)} - \{0\}$.

Set $u(0) = \tilde{u}(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})\, d\theta$.