Lecture 34 : More about the Dirichlet problem

On $D_1(0)$: Sol^n to D. prob with poly data is poly:

$$p(x,y) = p\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = \sum c_{nm} \underbrace{z^n \bar{z}^m}$$

$1 \quad n=m$

$z^{n-m} \quad n>m \qquad$ on $C_1(0)$

$\bar{z}^{m-n} \quad m>n$

$$= c_0 + P(z) + \overline{Q(z)}$$

Fun thing to do: Define harmonic fcns as complex valued fcns that satisfy the ave prop.

Step 1: Analytic polys satisfy ave prop:

At $z=0$: $\quad \frac{1}{2\pi}\int_0^{2\pi} P(e^{i\theta})\, d\theta$

$$= a_0 + \sum_{n=1}^{N} a_n \underbrace{\int_0^{2\pi} \underbrace{e^{in\theta}}_{(e^{i\theta})^n} d\theta}_{=0}$$

$$= a_0 = P(0) \checkmark$$

At $z=a$: $\quad P(z) = P((z-a)+a)$ expand

$= $ poly in $(z-a)$

$z=0$ argument works at $z=a$!

Ave prop $\Longrightarrow$ max princ.

Given continuous fcn $\varphi(\theta)$ on $C_1(0)$, Stone-Weierstraß

$\exists$ polys $p_n(x,y) \longrightarrow \varphi$ on $C_1(0)$.

[polys are an algebra that separates points.]

Hwk Prop: $p_n \rightarrow \varphi$ unif on $C_1(0)$.

$\Longrightarrow$ p_n unif Cauchy on $C_1(0)$.

Max princ $\Longrightarrow$ p_n unif Cauchy on $\overline{D_1(0)}$.

$\Longrightarrow$ p_n conv unif on $\overline{D_1(0)}$, say to u.

Ave prop for p_n $\Longrightarrow$ ave prop for u.

So u is harmonic! And $u = \varphi$ on $C_1(0)$.

<u>Next step</u>: Get Poisson formula.

Define $\quad K_N(z,w) = \bar{z}^N w^N + \dots + \bar{z} w + 1 + z\bar{w} + z^2\bar{w}^2 + \dots + z^N \bar{w}^N$

$$= 1 + \sum_{n=1}^{N} z^n \bar{w}^n + \sum_{n=1}^{N} \bar{z}^n w^n$$

Key $\quad 1, z^n, \bar{z}^n$ are $\perp$ on $C_1(0)$. Orthonormal!

$$\frac{1}{2\pi} \langle z^n, \bar{z}^m \rangle = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{e^{in\theta} \, \overline{e^{-im\theta}}}_{e^{i(n+m)\theta}} \, d\theta = 0$$

<u>Claim</u>: $\displaystyle \frac{1}{2\pi} \int_0^{2\pi} K_N(z, e^{i\theta}) \underbrace{(e^{i\theta})^n}_{z^n} \, d\theta = z^n$

Similarly if I put $\bar{z}^n$ or 1 inside $\int$.

So $$\frac{1}{2\pi}\int_0^{2\pi} K_N(z,e^{i\theta}) \underbrace{\begin{bmatrix} \text{poly} \\ \text{sol}^n \text{ to} \\ \text{D. Prob} \end{bmatrix}}_{\varphi(\theta)} d\theta = \begin{pmatrix} \text{poly sol}^n \\ \text{to} \\ \text{D. Prob} \end{pmatrix}$$

if $N > \deg \varphi$.

Can let $N \to \infty$.

$$K_N(z,w) = 1 + \underbrace{\sum_{n=1}^{N} z^n \bar{w}^n}_{z\bar{w}\sum_0^{N-1}(z\bar{w})^n} + \underbrace{\sum_{n=1}^{N} \bar{z}^n w^n}_{(\bar{z}w)\sum_0^{N-1}(\bar{z}w)^n}$$

$$= 1 + \frac{z\bar{w}\left(1-(z\bar{w})^N\right)}{1-z\bar{w}} + \frac{\bar{z}w\left(1-(\bar{z}w)^N\right)}{1-\bar{z}w}$$

$$\xrightarrow[\text{unif in } w]{z\in D_1(0) \text{ fixed}} \quad K(z,w) = \underbrace{1 + \frac{z\bar{w}}{1-z\bar{w}} + \frac{\bar{z}w}{1-\bar{z}w}}_{\text{the Poisson kernel!!}}$$

Know $$(\text{poly sol}^n) = \frac{1}{2\pi}\int_0^{2\pi} K_N(z,e^{i\theta})\, \overset{\text{poly}}{\varphi(\theta)}\, d\theta$$

$$\downarrow$$

$$\frac{1}{2\pi}\int_0^{2\pi} K(z,e^{i\theta})\,\varphi(\theta)\,d\theta$$

$$u = \frac{1}{2\pi}\int_0^{2\pi} K(z, e^{i\theta})\, \varphi(\theta)\, d\theta$$

not a poly

by taking limits, $p_n \to \varphi$ unif.

Poisson formula $\Longrightarrow$ u is C^∞ smooth and satisfies $\Delta u \equiv 0$.

<u>Fact</u>: On $D_1(0)$. φ rational $\Longrightarrow$ u rational!

<u>Why</u>: $R(x,y)$ rational and no singularities on $C_1(0)$.

$$R(x,y) = R\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = Q(z,\bar{z})$$

Hmmm. $\underset{\text{on } C_1(0)}{\bar{z} = \frac{1}{z}} = S(z) \leftarrow$ Schwarz fcn for $C_1(0)$.

$$R(x,y) = Q(z,\bar{z}) = \underbrace{Q\left(z,\frac{1}{z}\right)}_{\substack{\text{analytic} \\ \text{on } D_1(0) - \{\text{poles}\} \\ \text{and rational!}}} \quad \text{on } C_1(0)$$

$$\underset{\substack{\text{partial} \\ \text{fractions}}}{=} \underbrace{\sum\sum}_{a_k \in D_1(0)} \frac{A_{kn}}{(z-a_k)^n} + \underbrace{\sum\sum}_{|b_k|>1} \frac{B_{kn}}{(z-b_k)^n}$$

Take $z = \frac{1}{\bar{z}}$ here!

Get $R(x,y) = \underbrace{\begin{pmatrix}\text{Rational in } \bar{z} \\ \text{no poles in } \overline{D_1(0)}\end{pmatrix} + \begin{pmatrix}\text{Rational in } z \\ \text{no poles in } \overline{D_1(0)}\end{pmatrix}}_{\text{harmonic!}}$

Fact: Rational data $\Longrightarrow$ Rational sol^n to D. Prob on $D_1(0)$.

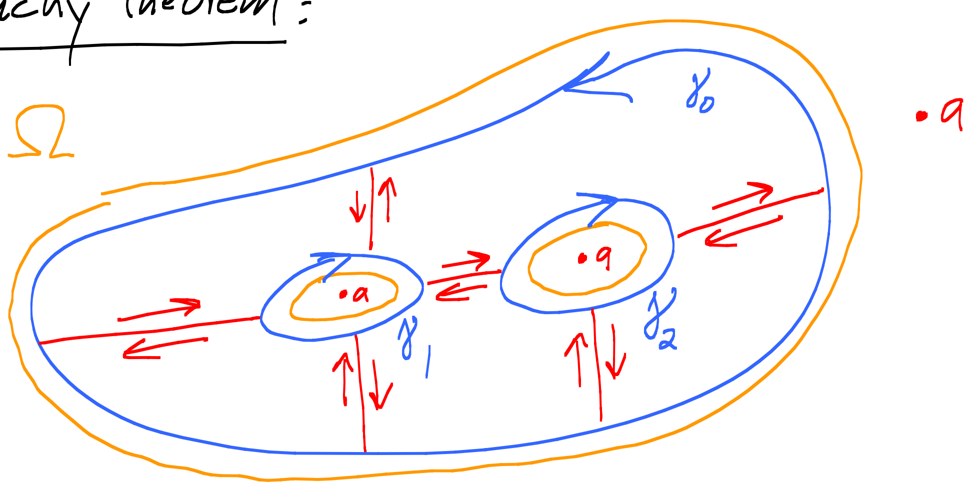
Thm: (B., Ebenfeld, Khavinson, Schapiro) Discs are the only domains with this feature!

Khavinson-Schapiro Conjecture: Poly data $\to$ Poly sol^{n}s to D. prob $\Longrightarrow$ Ω = disc or ellipse.

Rational $\to$ Rational sol^{n}s $\Longrightarrow$ Ω s. conn and Riemann map is rational.

General Cauchy Theorem:

f analytic on Ω



$0 = \int_\Gamma f \, dz$ where $\Gamma = \gamma_0 \cup \gamma_1 \cup \gamma_2$

Γ is called a "cycle"

Key $\underbrace{\mathrm{Ind}_\Gamma(a)}_{\sum \mathrm{Ind}_{\gamma_j}(a)} = 0$ for all $a \in \mathbb{C} - \Omega$

$$\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{\underbrace{z-a}_{f(z)}}\,dz = \left(\begin{array}{c}\#\text{ times } \gamma \\ \text{winds} \\ \text{around} \\ a\end{array}\right)$$

$f'(z)$

<u>Remark</u>: Cauchy Thm for annulus.