Lecture 35 General Cauchy Theorem

HWK 8 due Mon

Thm: Ω a domain, Γ a cycle in Ω such that

$$\mathrm{Ind}_\Gamma(a) = 0 \text{ for all } a \in \mathbb{C} - \Omega.$$

Then

1) $\int_\Gamma f \, dz = 0$ for analytic f and

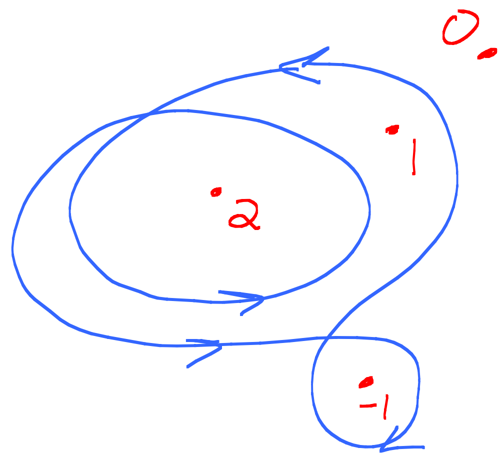
2) $\mathrm{Ind}_\Gamma(z_0) \cdot f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} \, dz, \quad z_0 \in \Omega - \mathrm{tr}(\gamma)$

Pf: (Ahlfors' book). Cycle $\Gamma = \gamma_1 \cup \gamma_2 \cup \dots \cup \gamma_N$ closed curves in Ω.

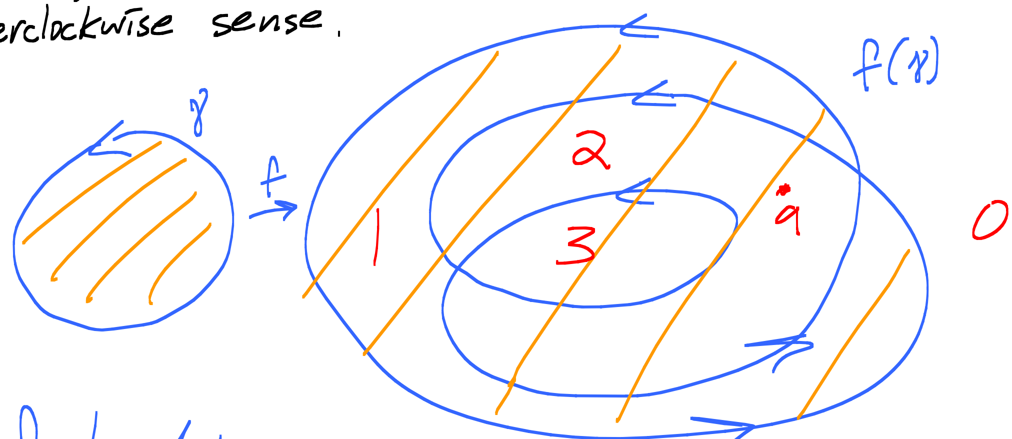
$$\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a} \, dz$$

$= \left(\text{winding number of } \gamma \text{ about } a \right)$

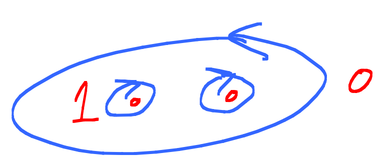
$=$ # times γ goes around a in counterclockwise sense.



Seen it before:



$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - a} \, dz = \frac{1}{2\pi i} \int_{f(\gamma)} \frac{1}{w-a} \, dw$$

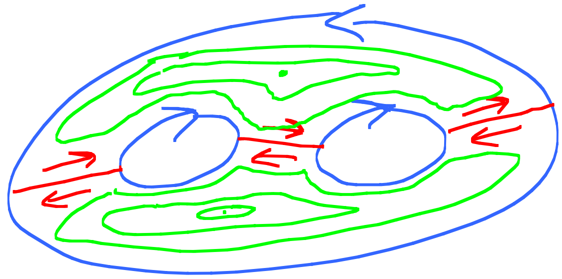


Def^n: $\mathrm{Ind}_\Gamma(a) = \sum_{n=1}^{N} \mathrm{Ind}_{\gamma_n}(a)$

Fancy word: $\mathrm{Ind}_\Gamma(a) = 0$ for all $a \in \mathbb{C} - \Omega$

Topology way to say this: Γ is homologous to zero in Ω: Write $\Gamma \sim 0$ or $\Gamma \sim_\Omega 0$.

Feeling:



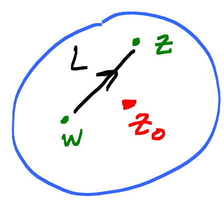
$\Gamma = \gamma_{top} \cup \gamma_{bottom}$

homotopic to a pt.

Pf:
$$g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\ f'(z) & z = w \end{cases}$$

Step 1: g is continuous on $\Omega \times \Omega$.

Clear at (z_0, w_0) when $z_0 \neq w_0$. Only need to check at (z_0, z_0).



← radius δ

$$\left| g(z,w) - g(z_0,z_0) \right| = \left| \underbrace{\frac{f(z)-f(w)}{z-w}} - \underbrace{f'(z_0)} \right|$$

$$= \frac{1}{z-w} \int_L f'(\zeta)\, d\zeta - f'(z_0) \cdot \frac{\int_L 1\, d\zeta}{z-w}$$

$$= \left| \frac{1}{z-w} \int_L \left(f'(\zeta) - f'(z_0) \right) d\zeta \right|$$

$$\leq \frac{1}{|z-w|}\underbrace{\left(\max_{L} \left| f'(\zeta) - f'(z_0)\right|\right)}_{\text{Can make } < \varepsilon \text{ by making } \delta>0 \text{ small enough because } f \text{ analytic} \Rightarrow f' \text{ continuous.}} \underbrace{\text{Length}(L)}_{|z-w|}$$

Step 2: Define $h(z) = \int_{\Gamma} g(z,w)\,dw$

Plan: Extend h to $\mathbb{C}$ as a bnd entire fcn!

$h \to 0$ at ∞. So $h \equiv 0$.

Step 3: h is analytic on Ω.

a) Clear that h is continuous on Ω.

[Why: g cont on $\Omega \times \Omega$. $tr(\Gamma) = \bigcup_{n=1}^{N} tr(\gamma_n)$ is compact. g unif cont on $\underbrace{\overline{D_\delta(z_0)} \times tr(\Gamma)}_{\text{compact}}$.]

b) $\Delta \subset \Omega$ a triangle.

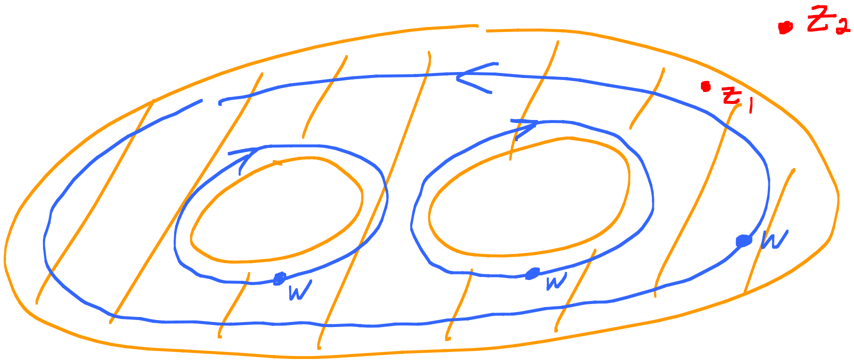
$$\int_{b\Delta} h\,dz = \int_{b\Delta}\left(\int_{\Gamma} g(z,w)\,dw\right)dz$$

$$\underset{\text{Fubini}}{=} \int_{\Gamma}\underbrace{\left(\int_{b\Delta} g(z,w)\,dz\right)}_{=0} dw = 0$$

because, for <u>fixed</u> w, $g(z,w)$ is analytic in z.

Morera's Thm $\Rightarrow$ h analytic.

<u>Step 4</u>: Extend h to $\mathbb{C}$



$$h(z) = \int_\Gamma \frac{f(z) - f(w)}{z - w}\, dw$$

$$= \underbrace{f(z)\int_\Gamma \frac{1}{z-w}\, dw}_{=0 \text{ for } z \text{ outside}} - \int_\Gamma \frac{f(w)}{z-w}\, dw$$

Aha! This is what h should be outside!

Define $\tilde{h}(z) = -\int_\Gamma \frac{f(w)}{z-w}\, dw$ on $\tilde{\Omega} = \{ z \in \mathbb{C} - tr(\gamma) : \mathrm{Ind}_\Gamma(z) = 0 \}$

<u>Claim</u>:

$$H(z) = \begin{cases} h(z) & z \in \Omega \\ \tilde{h}(z) & z \in \tilde{\Omega} - \Omega \end{cases}$$

Easy facts: $\tilde{\Omega}$ open.

$\mathbb{C} = \Omega \cup \tilde{\Omega}$

$\mathbb{C} - \Omega \subset \tilde{\Omega}$

Assume H entire for now.

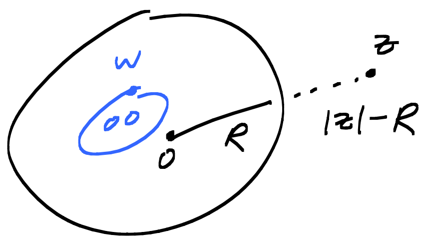
<u>Claim</u>: $H(z) \to 0$ as $z \to \infty$.

<u>Why</u>: $tr(\Gamma) \subset D_R(0)$.

Outside $C_R(0)$, $Ind_\Gamma \equiv 0$.

$$\text{So } H(z) = \underbrace{f(z) Ind_\Gamma}_{=0} - \int_\Gamma \frac{f(w)}{z-w}\,dw$$

$$\text{or } = -\int_\Gamma \frac{f(w)}{z-w}\,dw$$



$$|H(z)| \le \sum_{n=1}^{N} \frac{\left(\max_{\gamma_n} |f|\right)}{|z|-R} \cdot \text{Length}(\gamma_n)$$

$$\longrightarrow 0 \text{ as } z \to \infty.$$

Liouville's $\Rightarrow H \equiv 0$.

<u>Step 5</u> So $\displaystyle\int_\Gamma \frac{f(z)-f(w)}{z-w}\,dw = 0 \qquad z \in \Omega - tr(\gamma)$.

$$f(z) \cdot \int_\Gamma \frac{1}{z-w}\,dw = \int_\Gamma \frac{f(w)}{z-w}\,dw \quad \checkmark$$

Multiply by $\frac{1}{2\pi i}$.

<u>Step 6:</u> Cauchy Integral Formula

$\Longrightarrow$ Cauchy Theorem.