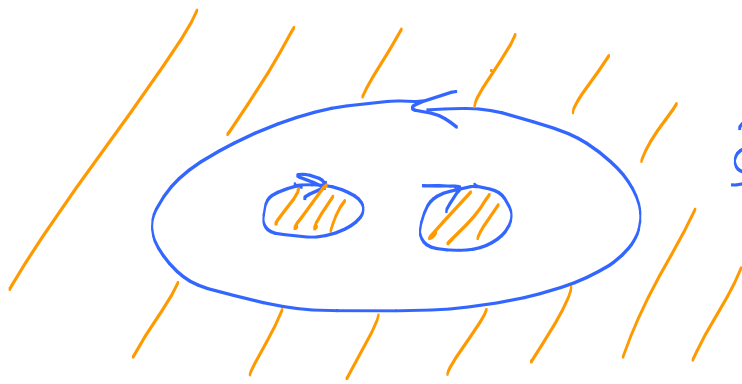
Lecture 36 General Residue Theorem

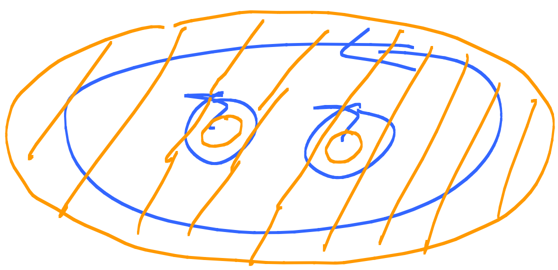
HWK 9 due Mon



$\tilde{\Omega} = \{ z \in \mathbb{C} - tr(\Gamma) : \mathrm{Ind}_{\Gamma}(z) = 0 \}$

$\tilde{\Omega}$ open $\subset \mathbb{C} - \Omega$

contains unbounded component of $\mathbb{C} - tr(\gamma)$.



Ω

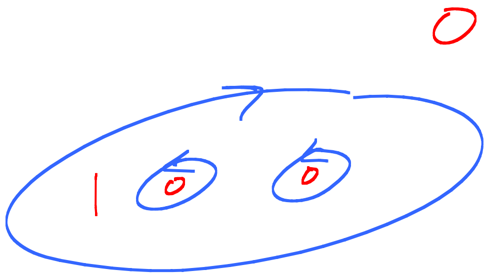
$$H(z) = \begin{cases} f(z) \int_{\Gamma} \frac{dw}{z-w} - \int_{\Gamma} \frac{f(w)}{z-w}\, dw & z \in \Omega \\ \underbrace{- \int_{\Gamma} \frac{f(w)}{z-w}\, dw}_{} & z \in \tilde{\Omega} \end{cases}$$

entire fcn!

analytic on $\tilde{\Omega}$ by diff under $\int$ as in HWK 2 : prob 1.

Things to know: $\int_{\Gamma} \frac{1}{z-w}\, dw$ is constant on connected components of $\mathbb{C} - tr(\Gamma)$ and $\to 0$ at ∞, so $= 0$ on unbounded component.

Why: $\dfrac{d}{dz}\displaystyle\int_\Gamma \frac{1}{z-w}\,dw = \int_\Gamma \frac{-1}{(z-w)^2}\,dw = \int_\Gamma \frac{d}{dw}\left[\frac{-1}{z-w}\right]dw = 0$



Γ is called simple because $\mathrm{Ind}_\Gamma(z) = 0$ or 1.

Think: "Inside" of simple cycle $= \{z : \mathrm{Ind}_\Gamma = 1\}$
"Outside" where $\mathrm{Ind}_\Gamma = 0$.

If f is analytic "inside and on" a simple cycle Γ, then $\displaystyle\int_\Gamma f\,dz = 0$.

Last time: Got $\dfrac{1}{2\pi i} f(z) \displaystyle\int_\Gamma \frac{1}{z-w}\,dw - \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{z-w}\,dw = 0$

"The Cauchy Integral Formula." ✓

C.I.F. $\Rightarrow$ Cauchy Theorem. Pf: Pick $z_0 \in \Omega - \mathrm{tr}(\Gamma)$.

Define $F(z) = (z - z_0) f(z)$.

C.I.F. for F at z_0:

$$0 = F(z_0)\,\mathrm{Ind}_\Gamma(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{\overbrace{f(z)(z-z_0)}^{F(z)}}{z - z_0}\,dz.$$

So $\displaystyle\int_\Gamma f\,dz = 0.$ ✓

New improved Cauchy Theorem for simp conn domains

f analytic on s.c. domain Ω and γ a closed curve in Ω.

1) $\int_\gamma f \, dz = 0$ ← not new

2) $f(z_0) \operatorname{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz$, $z_0 \notin tr(\gamma)$ (new)

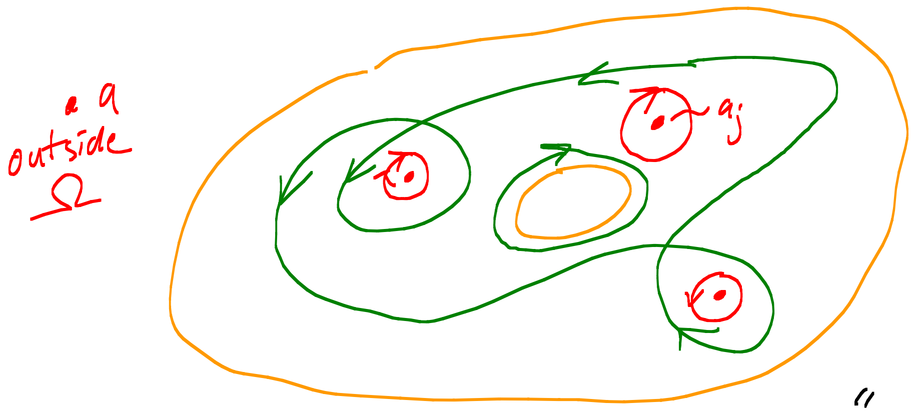
Pf: γ is homologous to zero because γ homotopic to a pt in Ω: $\int_\gamma \underbrace{\frac{1}{z-a}}_{\text{analytic on } \Omega} dz = \int_{pt} \frac{1}{z-a} dz = 0$, $a \in \mathbb{C} - \Omega$.

General Residue Theorem: Ω a domain, Γ a cycle in Ω such that $\operatorname{Ind}_\Gamma(a) = 0$ for all $a \in \mathbb{C} - \Omega$ ($\Gamma \sim 0$ in Ω), f analytic on Ω except at finitely many isolated singular points $A = \{a_1, a_2, \dots, a_N\}$ in Ω. Assume $A \cap tr(\Gamma) = \emptyset$.

Then $\int_\Gamma f \, dz = 2\pi i \sum_{n=1}^{N} \operatorname{Ind}_\Gamma(a_n) \operatorname{Res}_{a_n} f$

[Simple cycle : $\int_\Gamma f\,dz = 2\pi i \left(\begin{array}{c}\text{Sum of Res of } f \\ \text{"inside } \Gamma\text{"}\end{array} \right)$]

<u>Pf</u>: Let $\tilde{\Omega} - A$. Let $n_j = \mathrm{Ind}_\Gamma(a_j)$



Define $n_j C_\varepsilon(a_j)$:

$$z(t) = a_j + \varepsilon e^{i n_j t}$$

$$0 \le t \le 2\pi$$

"go around circle n_j times."

Can take $\varepsilon > 0$ so small that

1) $\overline{D_\varepsilon(a_j)} \cap \overline{D_\varepsilon(a_k)} = \emptyset \qquad j \neq k$

2) $\overline{D_\varepsilon(a_j)} \cap tr(\Gamma) = \emptyset \qquad$ all j

Note $\displaystyle\int_{mC_\varepsilon(a_j)} f \, dz = m \underbrace{\int_{C_\varepsilon(a_j)} f \, dz}_{= 2\pi i \, \mathrm{Res}_{a_j} f}$

Aha! $\tilde{\Omega} = \Omega - A$

$$\tilde{\Gamma} = \Gamma \cup \bigcup_{j}^{N} \left[-n_j C_\varepsilon(a_j) \right]$$

$n_j = \mathrm{Ind}_\Gamma(a_j)$

Note : Clear that $\mathrm{Ind}_{\tilde{\Gamma}}(a) = 0$ for $a \in \mathbb{C} - \Omega$

because $\mathrm{Ind}_{C_2(a_j)}(a) = 0$ for each a_j.

and now $\mathrm{Ind}_{\tilde{\Gamma}}(a_j) = 0$ too for each j.

$\tilde{\Gamma} \sim 0$ in $\tilde{\Omega}$.

General Cauchy $\Longrightarrow$

$$0 = \int_{\tilde{\Gamma}} f\,dz = \int_{\Gamma} f\,dz + 2\pi i \sum_{n=1}^{N} (-n_j)\,\mathrm{Res}_{a_j} f \quad \checkmark$$

<u>Remark</u>: Not really necessary to assume that A is finite, just discrete. $\leftarrow$ no limit pts.

[because $K = \{a \in \mathbb{C} : \mathrm{Ind}_\Gamma(a) \neq 0\} \cup \mathrm{tr}(\Gamma)$ is compact in Ω. If $A \cap K$ is infinite, it would have a limit pt in Ω. ↯.]

$\sum$ in Residue Thm would be finite. ✓

<u>Def^n</u>: If A is discrete and all singularities of f at pts in A are <u>poles</u>, then f is called <u>meromorphic</u> on Ω. [Think: give f the value ∞ at poles. Almost as good as analytic!]

For simple cycles, f analytic "inside and on", get Argument Principle, Rouché's Thm, etc.

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'}{f}\,dz = \left(\begin{array}{c}\#\text{ zeroes of } f\\ \text{inside } \Gamma\\ \text{counted with}\\ \text{mult}\end{array}\right)$$

f no zeroes on $tr(\Gamma)$

$= \#\text{zeroes} - \#\text{poles}$
if you allow poles off of $tr(\gamma)$.

$|f(z)-g(z)| < |f(z)|$ on $tr(\Gamma)$, then f and g have same # zeroes "inside Γ".

etc.