Lecture 37 Mittag-Leffler and Weierstraß Theorems

Mittag-Leffler Theorem: Given a seq $\{a_n\}_{n=1}^{\infty}$ of distinct pts in $\mathbb{C}$ such that $a_n \to \infty$ as $n \to \infty$ (= no limit pts in $\mathbb{C}$) and principal parts

$$R_n(z) = \frac{A_{N_n,n}}{(z-a_n)^{N_n}} + \cdots + \frac{A_{1,n}}{z-z_n},$$

there exists a meromorphic fcn f on $\mathbb{C}$ with poles at only the a_n's with princ part R_n at a_n [$f - R_n$ has a removable sing. at a_n].

Pf: Hmmm. First try: $\sum_1^\infty R_n(z)$

Taylor polynomial for $R_n(z)$

Second try: $\sum_1^\infty \underbrace{\left[R_n(z) - p_n(z)\right]}_{|\text{ }| < \frac{1}{2^n} \text{ on } D_{r_n}(0)}$

where $r_n = \frac{|a_n|}{2}$

Claim: $f(z) = \sum_1^\infty [R_n - p_n]$ solves prob!

Take big disc $D_R(0)$. $\exists N$ such that $r_n = \frac{|a_n|}{2} > R$ if $n > N$.

$$f(x) = \underbrace{\sum_{n=1}^{N}\left(R_n(z) - p_n(z)\right)}_{\substack{\text{rational fcn}\\ \text{with correct princ parts}\\ \text{in } D_R(0).}} + \underbrace{\sum_{n=N+1}^{\infty} \underbrace{\left(R_n(z) - p_n(z)\right)}_{|\text{ }| < \frac{1}{2^n}}}_{\substack{\text{Weierstraß M-test}\\ \Longrightarrow \text{ converges unif to}\\ \text{analytic fcn on } D_R(0).}}$$

Remark: M-L Thm true on any domain Ω, $A=\{a_n\}_{n=1}^{\infty}$ a discrete subset. Lars Hörmander: Several Complex Vars, Chap 1.

Weierstraß: Given a discrete seq $\{a_n\}$ as in M-L Thm and positive integers m_n, there is an entire fcn having zeroes of order m_n at a_n and no other zeroes.

Classic proof: $f(z) = \prod_{n=1}^{\infty} \underbrace{\left(1-\frac{z}{a_n}\right)^{m_n} e^{P_n(z)}}$ ← Weierstraß polys

need terms → 1 fast enough to converge unif on compacts → sol^n

My proof: M-L ⟹ Weirstraß via general Res. Thm.

Hmmm. $\frac{f'}{f} = \underbrace{\frac{m_n}{z-a_n} + (\text{analytic near } a_n)}$

mero fcn. poles at a_n. P.P. = $\frac{m_n}{z-a_n}$

Hmmm. $f(z) = e^{\log f} = e^{\int \frac{f'}{f}\,dw}$

Aha! "Define" $f(z) = \exp\left(\int_{\gamma_a^z} F(w)\,dw\right)$

where F is a fcn given by M-L Thm with poles at each a_n with Princ part $\frac{m_n}{z-a_n}$ there, and γ_a^z is a curve that starts at some fixed $a \in \mathbb{C}-\{a_n\}_{n=1}^{\infty}$ and ends at $z \in \mathbb{C}-\{a_n\}_{n=1}^{\infty}$ and avoids all the a_n.

<u>Step 1</u>: $f(z)$ is well defined.

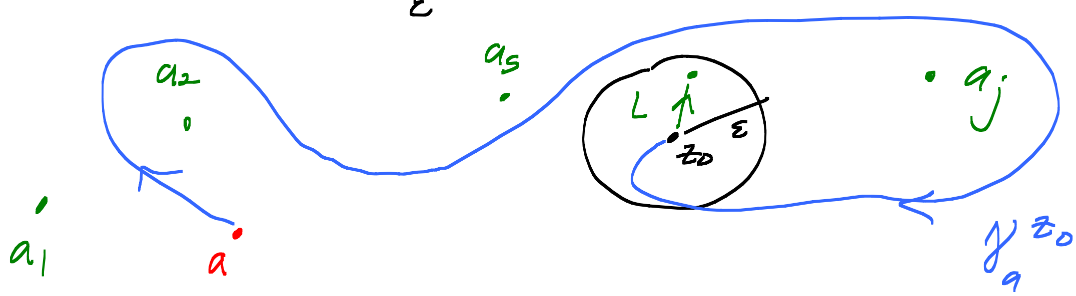
$$\frac{f(z)}{\tilde{f}(z)} = \frac{\exp\left(\int_{\gamma_a^z} F\,dw\right)}{\exp\left(\int_{\tilde{\gamma}_a^z} F\,dw\right)} = \exp\left(\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z}\right) F\,dw\right)$$

$\Gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$ closed curve

$$= \exp\Big(\underbrace{2\pi i \sum \mathrm{Ind}_\Gamma(a_n)}_{\text{integer times } 2\pi i\,!} \cdot \underbrace{m_n}_{\mathrm{Res}_{a_n} F}\Big) = 1$$

<u>Step 2</u>: f is analytic on $\mathbb{C} - A$.

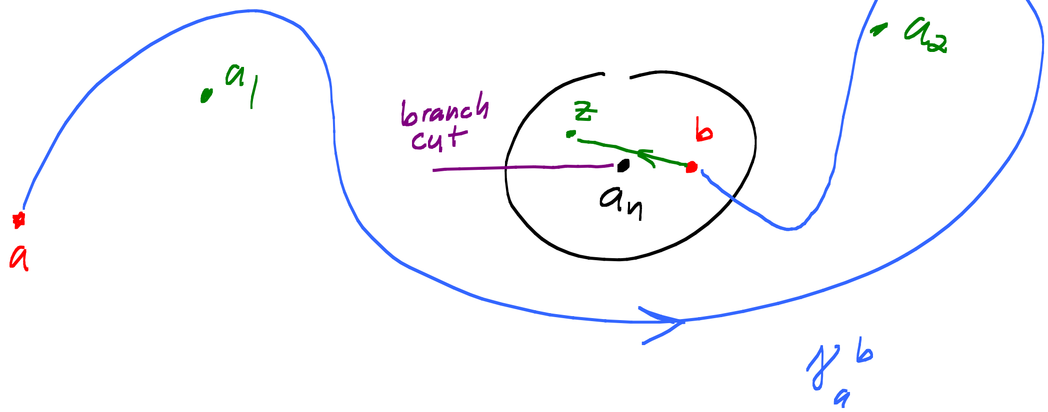
Restrict attn to $D_\varepsilon(z_0) \subset \mathbb{C} - A$.



$$f(z) = \exp\Big(\underbrace{\int_{\gamma_a^{z_0}} F\,dw}_{\text{const}} + \underbrace{\int_{L_{z_0}^z} F\,dw}_{\text{diff'ble and its deriv is } F}\Big)$$

and furthermore $f'(z) = f(z)\,F(z)$. $\left[\ \frac{f'}{f} = F\ \right]$

Step 3: a_n is a removable sing for f.



$$f(z) = \exp\left(\underbrace{\int_{\gamma_a^b} F\,dw}_{\text{const}} + \underbrace{\int_{L_b^z} \frac{m_n}{z-a_n} + \underbrace{H(w)}_{\text{analytic near } a_n}\,dw}_{F(z)} \right)$$

$$= \exp\Big(c + m_n \operatorname{Log}(z-a_n) - m_n \operatorname{Log}(b-a_n) + h(z) \Big)$$

$$= \underbrace{\exp\big(m_n \operatorname{Log}(z-a_n)\big)}_{(z-a_n)^{m_n}} \exp\big(\text{analytic stuff}\big)$$

(Analytic above and below cut and cont. up to cut and agree. Cut is removable.)

Aha! a_n is a zero of order m_n. Done!

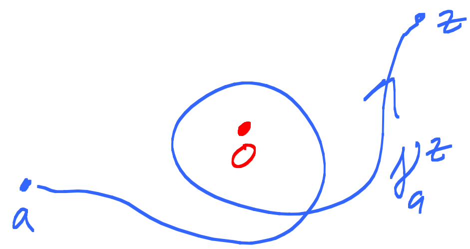
Cor: Combine M-L and W, get that we can prescrible finitely many terms in power series (or the Laurent expansion) on a discrete set of points.

Cor: Meromorphic fcns = Quotients of entire fcns (denom $\not\equiv 0$).

Pf: Suppose f is mero on $\mathbb{C}$. Poles at $\{a_n\}$ discrete of order m_n. Get g from Weir. with zeroes of order m_n at a_n.

$g \cdot f$ has removable sing at each a_n, so it is equal to entire fcn h on $\mathbb{C} - A$. $f = \frac{h}{g}$ ✓

HW9: 1



Cube root of $f(z)$: Try $F(z) = \exp\left(\frac{1}{3}\int_{\gamma_a^z} \frac{f'}{f}\, dw\right)$

Show F well defined.

Want $F^3 = c f$. Trick: $\frac{d}{dz}\left(\frac{F^3}{f}\right) \equiv c$.