# Lecture 38 Infinite products

HWK 9 due Mon

$$\prod_{n=1}^{\infty} p_n = \lim_{N\to\infty} \prod_{n=1}^{N} p_n \quad \text{where } p_n \neq 0, \forall n.$$

Lemma: $\prod p_n \to L \neq 0$, then $p_n \to 1$ as $n \to \infty$.

Pf: $p_N = \dfrac{\prod^N p_n}{\prod^{N-1} p_n} \longrightarrow \dfrac{L}{L} = 1$ as $N \to \infty$

Traditional to write $\prod_{n=1}^{\infty} (1 + a_n)$, $(a_n \to 0)$.

Lemma: $\prod_{n=1}^{\infty} (1+a_n)$ converges to $L \neq 0$

$$\iff \sum_{n=1}^{\infty} \operatorname{Log}(1+a_n) \text{ converges.}$$

Pf: $\Leftarrow$ easy $\displaystyle\prod_{n=1}^{N} (1+a_n) = e^{\sum^N \operatorname{Log}(1+a_n)}$.

$\Rightarrow$ Read Stein p. 141.

Fact: $\operatorname{Log}(1+z)$ analytic on $D_1(0)$ and has a simple zero at $z=0$. And $\frac{d}{dz}\operatorname{Log}(1+z)\Big|_{z=0} = 1$.

$$(1-\varepsilon)|a_n| \leq |\operatorname{Log}(1+a_n)| \leq (1+\varepsilon)|a_n| \quad (*)$$

Given $\varepsilon > 0$, $\exists\, \delta > 0$ such $(*)$ holds when $|a_n| < \delta$.

**Def^n:** $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely when

$$\sum_{n=1}^{\infty} |a_n| < \infty.$$

**Fact:** If $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly on compact subsets of a domain $\Omega$, $f_n$ analytic, then $\prod_{n=1}^{\infty}(1+f_n(z))$ converges absolutely and uniformly on compact subsets of $\Omega$ to an analytic fcn.

**EX:** $\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = f(z)$

$$\sum_{n=1}^{\infty}\left|\frac{z^2}{n^2}\right| = |z|^2 \underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{\frac{\pi^2}{6}} < \infty$$

$(f \text{ entire})$

Near $z \in \mathbb{Z}$, pull relevant factor out in front to see $f$ has simple zeroes at pts in $\mathbb{Z}$.

**EX:** $\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)$. Ouch! $\sum_{1}^{\infty}\left|\frac{z}{n}\right| = |z| \underbrace{\sum_{1}^{\infty}\frac{1}{n}}_{=\infty}$

Weierstraß: $f$ to have zeroes at $a_n$, $a_n \to \infty$.

$$\prod (z - a_n)$$

$$\prod \left(1 - \frac{z}{a_n}\right)$$ (factor to make term closer to one on bigger and bigger compact sets as $a_n \to \infty$)

$$1 = (1-z)\, e^{-\mathrm{Log}(1-z)}$$

$$-\mathrm{Log}(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

$$P_N(z) = z + \frac{z^2}{2} + \cdots + \frac{z^N}{N}$$

$(1-z)\, e^{P_N(z)}$ as close 1 as we want on $D_{1/2}(0)$ by taking $N$ big.

$\displaystyle\prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{a_n}\right) e^{P_{N_n}\left(\frac{z}{a_n}\right)}\right]^{m_n}$ entire and has zeroes at $a_n$ with order $m_n$ if I choose $N_n \to \infty$ properly.

Famous formulas: $\displaystyle \sin \pi z = \pi z \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(1 - \frac{z}{n}\right) e^{z/n}$

$$= \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

Why: Step 1: $\displaystyle\sum \frac{1}{(z-n)^2} = \sum_{-N}^{N} \frac{1}{(z-n)^2} + (\text{Tail end})$

terms $\sim \dfrac{1}{(m-N)^2}$

Weierstraß M-test.

$\rightarrow$ analytic

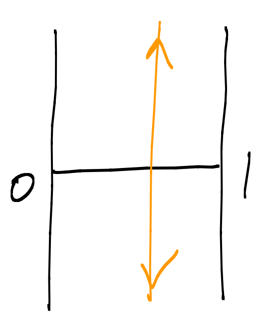
So $\sum$ is meromorphic on $\mathbb{C}$ with double poles at $n \in \mathbb{Z}$.

Step 2: $\displaystyle\frac{\pi^2}{\sin^2 \pi z} = \underbrace{\frac{1}{(z-n)^2}}_{\text{Princ part.}} + (\text{analytic})$ near $n$

Step 3: Both sides are periodic with period $1$.

So $H(z) = (\text{left}) - (\text{right})$ is entire and periodic.

Step 4:




$H(z) \to 0$ along vertical lines

Get $H$ bndd on strip!

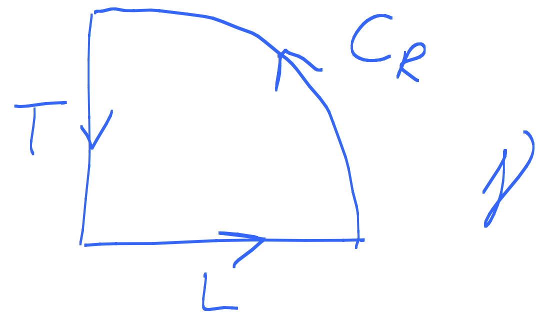
Periodic $\Rightarrow$ $H$ bndd entire.

Step 5: Liouville's $\Rightarrow$ const. $\to 0$ as $\uparrow$ $\Rightarrow$ const $= 0$. ✓

## HWK 9 hint:

$$P(z) = z^{1998} + z + 2001$$

Argument principal

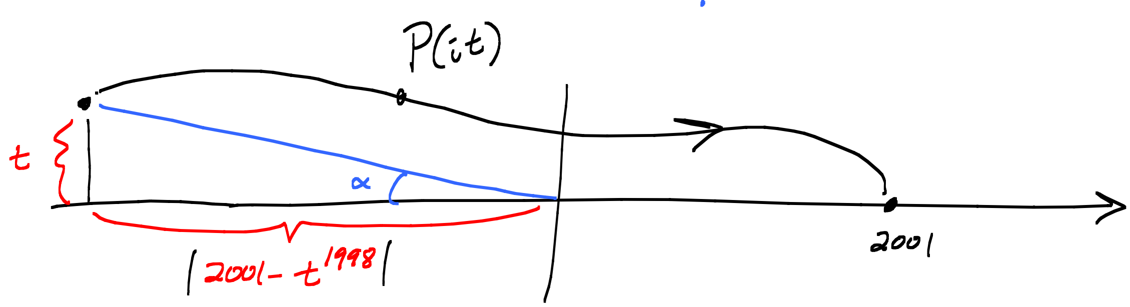



$$2\pi \left( \begin{array}{c} \#\text{ zeroes of } P \\ \text{in } \gamma \\ \text{(with mult)} \end{array} \right) = \frac{1}{i} \int_\gamma \frac{P'}{P}\, dz = \Delta \arg_\gamma P$$

$\Delta \arg$ on $L = 0$.

$\Delta \arg$ on $T$: $\quad P(it) = (2001 - t^{1998}) + it$

stays in UHP




$\alpha \to 0$ as $t \to \infty$

Key fact for $C_R$ part: (Lecture 21)

$$\underbrace{\int_{C_R} \frac{P'}{P}\, dz}_{\to \text{ limit as } R \to \infty} = \left( \begin{array}{c} \text{real} \\ \text{part} \end{array} \right) + i\, \Delta \arg_{C_R} P$$

<u>Def$^n$</u>: $f$ analytic on $\{z : |z| > R\}$.

We say $f$ has the singularity at $\infty$ that $f(\frac{1}{z})$ has at $z=0$.

<u>Facts</u>: 1) Entire fcn with a pole at $\infty$ is a poly.

2) Meromorphic fcn with finitely many poles and a pole or a removable sing at $\infty$ must be rational.