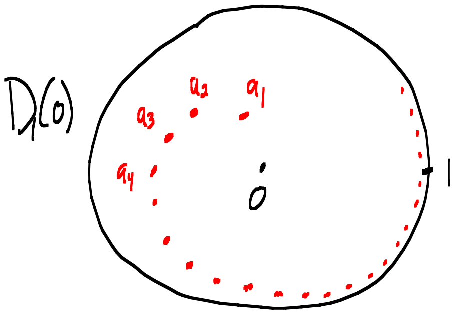
Review lecture 1

Final Exam Wed., May 1, 8:00-10:00 am in LAWSN B151

M-L on simply connected $\Longrightarrow$ Weierstraß Thm about zeroes on simply connected domains by proof I gave $\Omega = \mathbb{C}$.

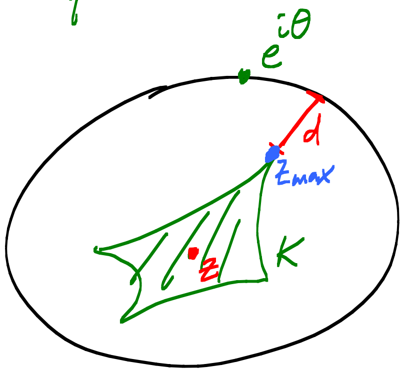
(Weierstraß Thm about zeroes true on <u>any</u> domain in $\mathbb{C}$)



a_n spiraling out to $|z|=1$, limit points dense in $C_1(0)$. Get fcn f with zeroes at a_n's. Cannot be extended analyticly to any bigger open set. Domains in $\mathbb{C}$ are "domains of holomorphy".

4. *(30 pts.)* Suppose that $f_n(z)$ is a sequence of functions that are continuous on $\overline{D_1(0)}$, analytic on $D_1(0)$, and such that $\int_0^{2\pi} |f_n(e^{i\theta})|\, d\theta < 1$ for all n. Prove that there is a subsequence that converges uniformly on compact subsets of $D_1(0)$.

Montel's Thm! Need to show $\{f_n\}_1^\infty$ is uniformly bdd on compact subsets of $D_1(0)$.



$$\left|f_n(z)\right| = \left|\frac{1}{2\pi i}\int_{C_1(0)} \frac{f_n(w)}{w-z}\, dz\right|$$

$$= \left|\frac{1}{2\pi i}\int_0^{2\pi} f_n(e^{i\theta}) \frac{1}{e^{i\theta}-z}\, i e^{i\theta}\, d\theta\right|$$

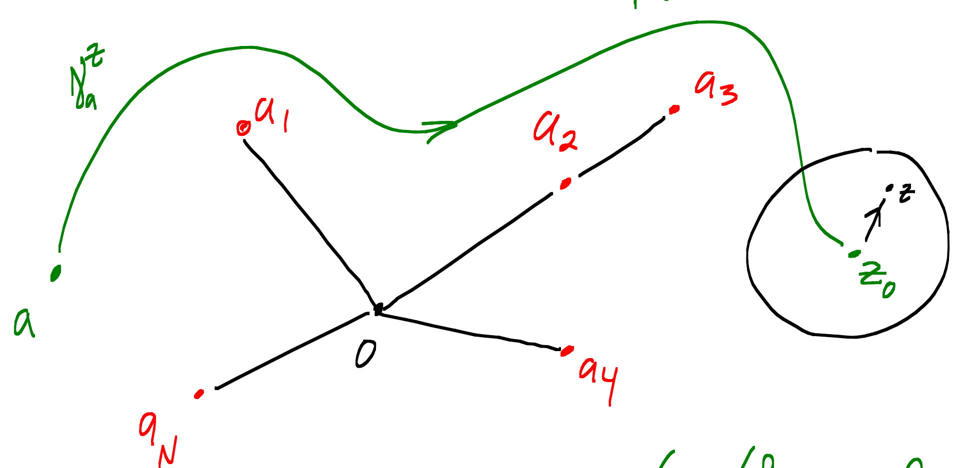
$$\le \frac{1}{2\pi}\int_0^{2\pi} \left|f_n(e^{i\theta})\right| \frac{1}{d}\, d\theta$$

Bdd ✓

$|z|$ is continuous on compact K. Has max at $z_{max} \in K$. $d = 1 - |z_{max}| > 0$

5. *(40 pts.)* Suppose $a_1, a_2, \ldots, a_N$ are distinct non-zero complex numbers and let Ω denote the domain obtained from $\mathbb{C}$ by removing each of the closed line segments joining a_k to the origin, $k = 1, \ldots, N$. Prove that there is an analytic function f on Ω such that

$$f(z)^N = \underbrace{\prod_{k=1}^{N}(z - a_k)}_{F(z)}.$$



Try

$$f(z) = \exp\left(\frac{1}{N}\int_{\gamma_a^z} \frac{F'(w)}{F(w)}\,dw\right)$$

f well define:

$$\frac{f(z)}{\tilde{f}(z)} = \exp\left(\frac{1}{N}\underbrace{\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a}\right)\frac{F'}{F}\,dw}_{2\pi i \sum^{N} \underbrace{\mathrm{Ind}_\Gamma(a_n)}_{\text{all the same, say } m}\, \underbrace{\mathrm{Res}_{a_n}\frac{F'}{F}}_{=1\,=\,\text{order of zero at } a_n} \,=\, 2\pi i N m}\right) = e^{2\pi i m} = 1 \checkmark$$

$\Gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$

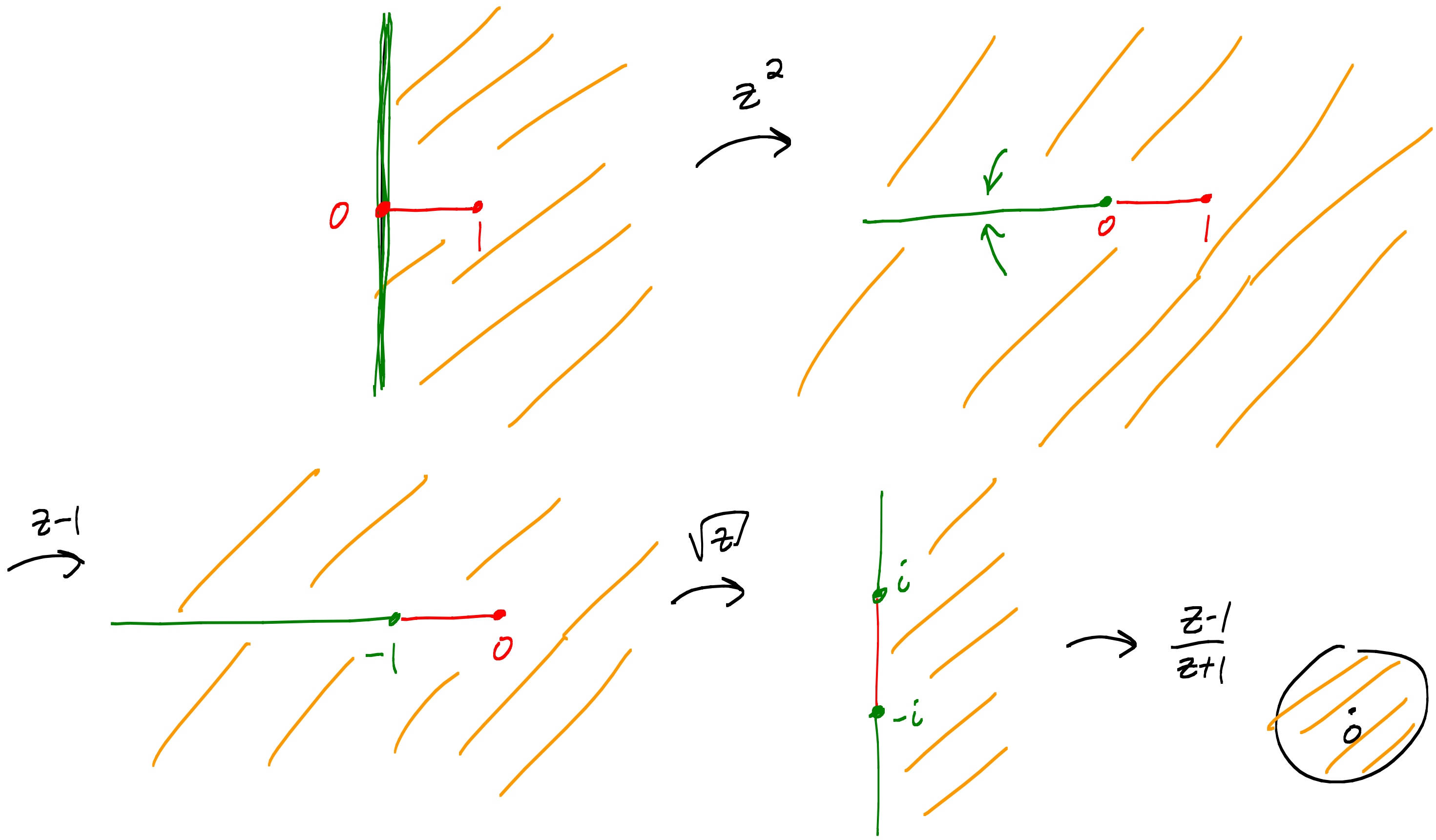
closed

Next: $f' = \underbrace{\exp\left(\frac{1}{N}\int_{\gamma_a^z}\frac{F'}{F}\,dw\right)}_{f} \cdot \frac{1}{N}\frac{F'}{F}$ So $\frac{f'}{f} = \frac{1}{N}\frac{F'}{F}$

Last step: $\left(\frac{f^N}{F}\right)' \equiv 0$ because of

So $f^N = cF$. Correct $f_{Cor} = \frac{1}{\sqrt[N]{c}} f$.

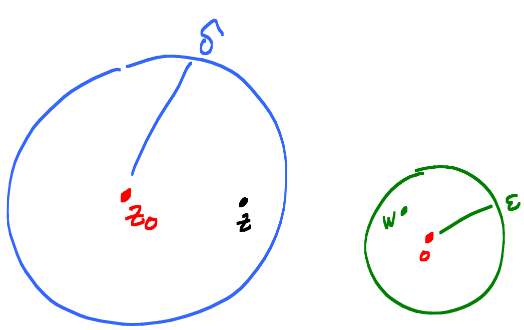
3. *(30 pts.)* Find a one-to-one conformal mapping from the region $\{z : \text{Re } z > 0\} - (0, 1]$ onto the unit disc.



6. *(40 pts.)* Suppose that $f(z)$ is an analytic functions with a zero of order N at z_0. Prove that there exist $\epsilon > 0$ and $\delta > 0$ such that, for every $w \in \mathbb{C}$ with $0 < |w| < \epsilon$, the equation $f(z) = w$ has exactly N *distinct* roots in $D_\delta(z_0)$.

Hint: Rouché's Theorem.

f has a zero of order N at z_0



$f(z) = w$ has N distinct roots in $D_\delta(z_0)$

Step 1: Shrink δ so that z_0 is the only zero of f in $\overline{D_\delta(z_0)}$

Step 2: Think Rouché: $f = f$, $g = f - w$

$|f-g| = |w| \le m < |f|$ on $C_\delta(z_0)$.

↑ want

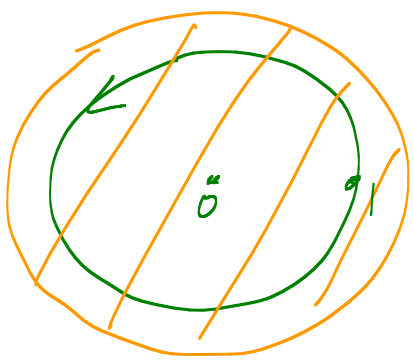
$m = \min |f|$ on $C_\delta(z_0)$

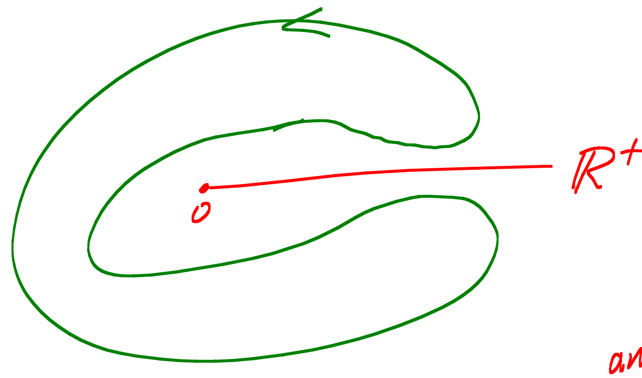
Get $f(z)-w$ has N zeroes.

Take $\varepsilon = m$.

Aha! f' has a zero of order $N-1$ at z_0. Shrink δ some more so z_0 is only zero of f' in $\overline{D_\delta}$ too. N zeroes, all simple $\Rightarrow$ distinct!

2. *(30 pts.)* Suppose that f is analytic in a neighborhood of the closed unit disc and that $f(z)$ is never in the set $\{x \in \mathbb{R} : x \geq 0\}$ when $|z| = 1$. Show that f has no zeroes in the unit disc.



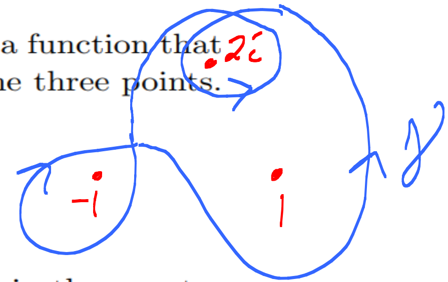


analytic on nbhd of $C_1(0)$

Arg Princ.: $\left(\begin{array}{c}\#\text{ zeroes of } f \\ \text{in } C_1(0)\end{array}\right) = \dfrac{1}{2\pi i}\displaystyle\int_{C_1(0)} \frac{f'}{f}\,dz = \frac{1}{2\pi i}\int_{C_1(0)} \frac{d}{dz}\,\underbrace{\log f}\,dz$

$= 0$

where $\log w = \operatorname{Ln}|w| + i\theta$ where $\theta \in \arg w$ with $0 < \theta < 2\pi$



6. *(30 pts.)* Suppose that $a_1 = -1$, $a_2 = 1$, and $a_3 = 2i$ and that f is a function that is analytic on $\mathbb{C} - \{a_1, a_2, a_3\}$ that has essential singularities at the three points. Suppose also that

$$\int_{C_1(a_n)} f\,dz = \sqrt{n} \quad \text{for } n = 1, 2, 3,$$

where $C_1(z_0)$ denotes the circle of radius 1 about z_0 parametrized in the counter clockwise sense. Draw a closed curve γ such that

$$\mathrm{Ind}_\gamma a_1 = -1, \qquad \mathrm{Ind}_\gamma a_2 = 1, \quad \text{and} \quad \mathrm{Ind}_\gamma a_3 = 2.$$

Explain how to define a cycle Γ so that the General Cauchy Theorem on the domain $\Omega = \mathbb{C} - \{a_1, a_2, a_3\}$ can be used to compute

$$\int_\gamma f\,dz.$$

Find the value of the integral and explain your reasoning. (You are not allowed to use the General Residue Theorem here. If you failed to draw such a γ, you may assume that such a γ exists.)