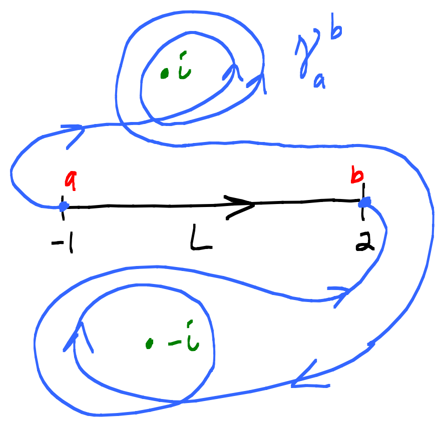
Review 2 Final Exam Wednesday, May 1, 8:00-10:00am, LWSN B151

Office Hours: M, T 2:00-3:00 pm

5. What are the possible values of the integral

$$\int_\gamma \frac{z}{z^2+1}\,dz,$$

where γ is a path that starts at $z=-1$ and ends at $z=2$ without passing through the points $\pm i$. Explain.



$$\frac{1}{2}\int_{-1}^{2} \frac{2t}{t^2+1}\cdot 1\,dt = A$$

$\Gamma = L \cup (-\gamma_a^b)$ is a closed curve (cycle)

Gen Res Thm $\underbrace{\int_\Gamma}_{\int_L - \int_{\gamma_a^b}} \frac{z}{z^2+1}\,dz = 2\pi i \sum \mathrm{Ind}_\Gamma(\pm i)\,\mathrm{Res}_{\pm i} \frac{z}{z^2+1}$

$$\underbrace{\int_L}_{=A} - \int_{\gamma_a^b} = 2\pi i\left(n_1 \frac{i}{2i} + n_2 \frac{(-i)}{2(-i)}\right)$$

$$\int_{\gamma_a^b} \frac{z}{z^2+1}\,dz = A + \pi i m, \quad m \in \mathbb{Z}$$

5. Let $a_0 = 0$ and $a_1 = 1$. The Fibonacci numbers are defined inductively via

$$a_{n+2} = a_{n+1} + a_n \qquad \text{for } n = 0, 1, 2, \ldots$$

Find the radius of convergence of the power series $\sum a_n z^n$. Hint: Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and compare the power series for $z^2 f(z)$, $zf(z)$, and $f(z)$. Find a closed form formula for f. When you get the picture, make sure everything you say is true. (For example, don't say that $f(z)$ is defined someplace until you know it is.)

$$\begin{array}{lrrl} z^2 f(z) = & & & a_0 z^2 + a_1 z^3 + a_2 z^4 + \cdots \\ z f(z) = & & a_0 z + & a_1 z^2 + a_2 z^3 + a_3 z^4 + \cdots \\ f(z) = & a_0 + & a_1 z + & a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \end{array}$$

$$\left(f(z) - \overset{0}{a_0} - \overset{1}{a_1} z\right) = \left(z f(z) - \overset{0}{a_0} z\right) + z^2 f(z)$$

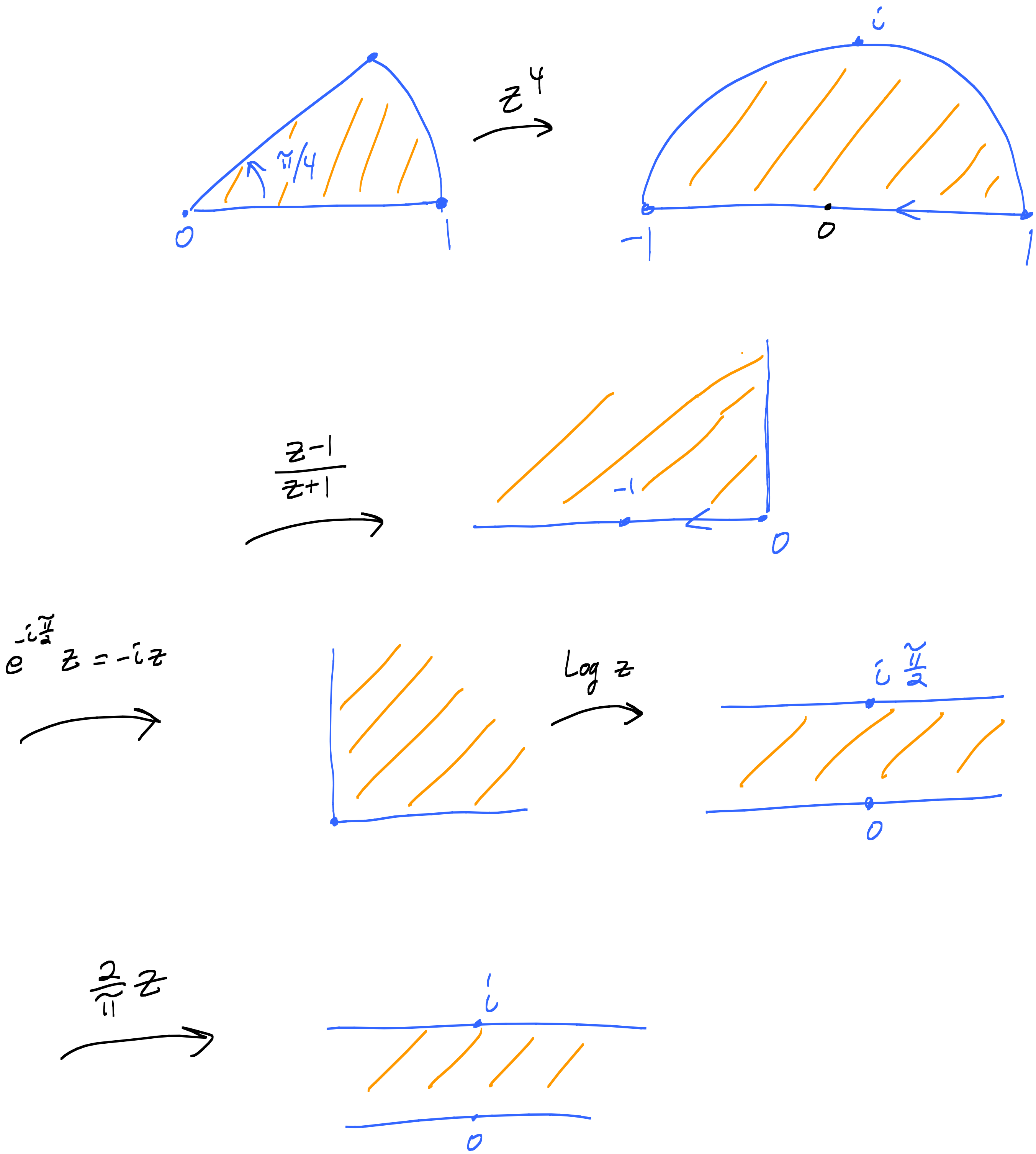
<u>If</u> power series converges near 0, then

$$f(z) = \frac{-z}{\underbrace{z^2 + z - 1}}$$

Aha!

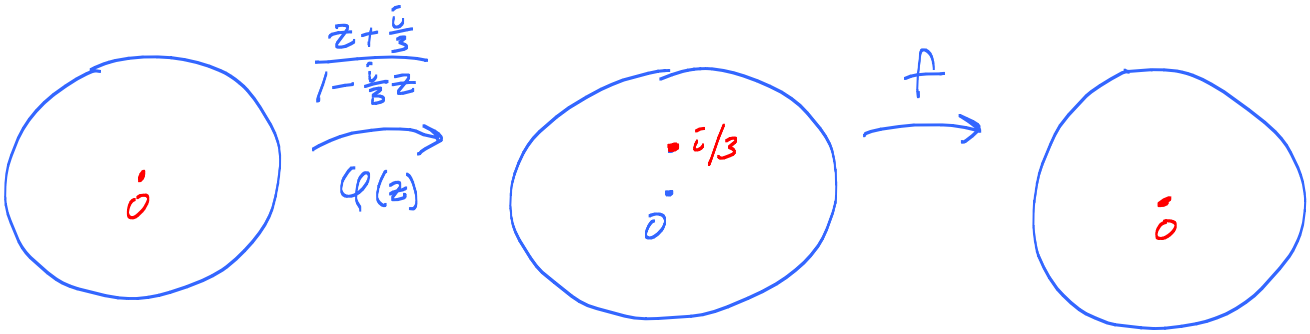
R. of Conv. of power series for this = dist from origin to closest zero of denom

4. Find a one-to-one conformal mapping of the "piece of pie" $\{re^{i\theta} : 0 < r < 1,\ 0 < \theta < \pi/4\}$ onto the horizontal strip $\{z : 0 < \operatorname{Im} z < 1\}$.



1. *(30 pts.)* Let $\mathcal{F}$ denote the family of all analytic functions f that map the unit disc into itself with $f(i/3) = 0$. Find

$$M = \sup\{\mathrm{Im}\ f(0) : f \in \mathcal{F}\}$$



Schwarz: $\quad |f(\underbrace{\varphi(z)}_{=0})| \leq |z|$

$$z = \varphi^{-1}(0) = \frac{0 - \frac{i}{3}}{1 + \frac{i}{3}\cdot 0} = \frac{i}{3}$$

So $\quad |f(0)| \leq \left|\frac{i}{3}\right| = \frac{1}{3}$

$$\mathrm{Im}\, f(0) \leq |\mathrm{Im}\, f(0)| \leq |f(0)| \leq \frac{1}{3}$$

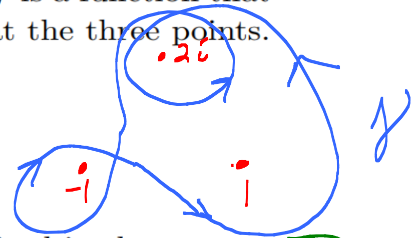
Extremal for $|f(0)|$: $\quad f(\varphi(z)) = \lambda z \qquad |\lambda| = 1$

$$f(z) = \lambda\, \varphi^{-1}(z) = \lambda\, \frac{z - \frac{i}{3}}{1 + \frac{i}{3}z}$$

Aha! Can choose λ so that $f(0) = \lambda\left(-\frac{i}{3}\right)$ points in pos. Imag. direction. $\lambda = i$.

$$M = \frac{1}{3}$$

6. *(30 pts.)* Suppose that $a_1 = -1$, $a_2 = 1$, and $a_3 = 2i$ and that f is a function that is analytic on $\mathbb{C} - \{a_1, a_2, a_3\}$ that has essential singularities at the three points. Suppose also that

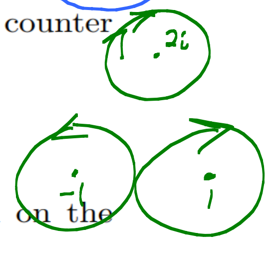


$$\int_{C_1(a_n)} f\,dz = \sqrt{n} \quad \text{for } n = 1, 2, 3,$$

$= 2\pi i \operatorname{Res}_{a_n} f$

where $C_1(z_0)$ denotes the circle of radius 1 about z_0 parametrized in the counter clockwise sense. Draw a closed curve γ such that

Add curves:



$$\operatorname{Ind}_\gamma a_1 = -1, \qquad \operatorname{Ind}_\gamma a_2 = 1, \quad \text{and} \quad \operatorname{Ind}_\gamma a_3 = 2.$$

Explain how to define a cycle Γ so that the General Cauchy Theorem on the domain $\Omega = \mathbb{C} - \{a_1, a_2, a_3\}$ can be used to compute

$$\int_\gamma f\,dz. = 2\pi i \sum_{n=1}^{3} \operatorname{Ind}_\gamma(a_n) \operatorname{Res}_{a_n} f$$

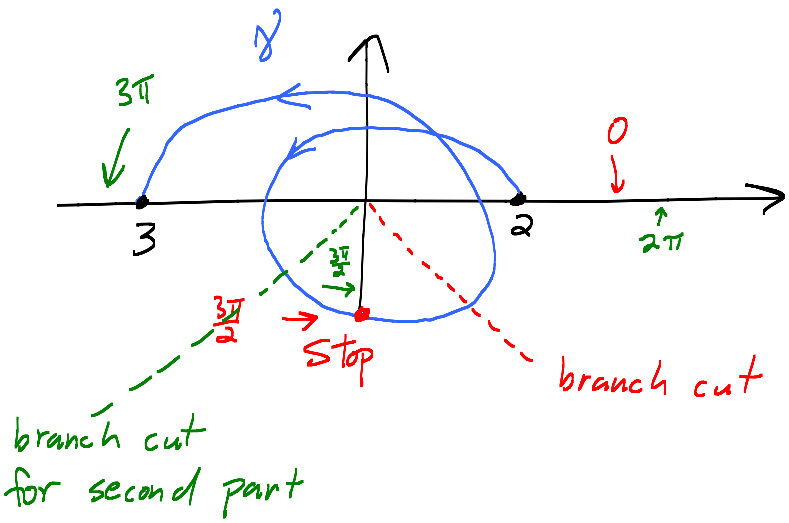
$\Gamma = \gamma$ plus circles is ~ 0 in $\Omega = \mathbb{C} - \{a_n\}_1^3$

Find the value of the integral and explain your reasoning. (You are not allowed to use the General Residue Theorem here. If you failed to draw such a γ, you may assume that such a γ exists.)

1. *(40 pts.)* Use branches of the complex log function to compute $\int_\gamma \frac{1}{z}\,dz$ if γ is a path parametrized by

$$z(t) = r(t)e^{it}, \qquad 0 \le t \le 3\pi,$$

where $r(t)$ is a positive real valued function such that $r(0) = 2$ and $r(3\pi) = 3$. Explain how you arrived at your answer.



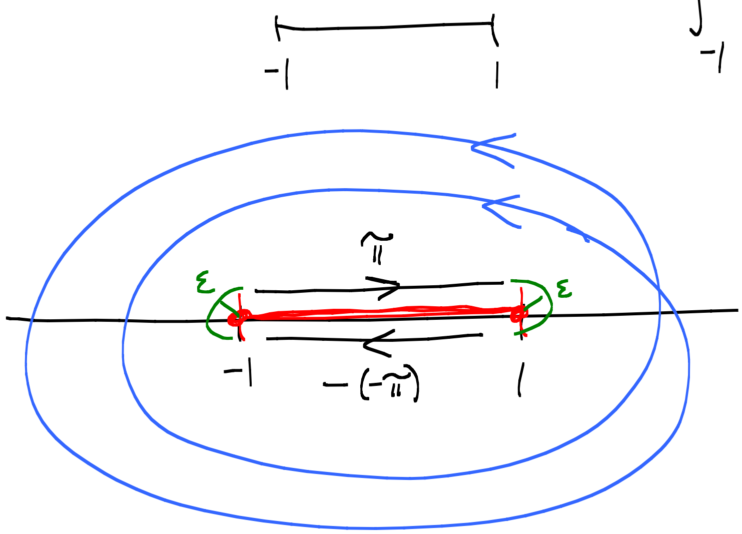
Ans: $(\ln 3 - \ln 2) + i\,3\pi$

6. Show that a single valued analytic branch of $\sqrt{1-z^2}$ can be defined on $\mathbb{C}-[-1,1]$. What are the possible values of

$$\int_\gamma \frac{dz}{\sqrt{1-z^2}}$$

when γ is a closed curve in $\mathbb{C}-[-1,1]$.

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\,dt = \operatorname{Arc Sin} t \Big|_{-1}^{1} = \tilde{\pi}$$



Think

$$\sqrt{1-z^2} = \sqrt{1-z}\,\sqrt{1+z}$$

$$\frac{f'}{f} \qquad f = 1-z^2$$

or Define $\quad c\exp\left(\frac{-1}{2}\int \frac{(-2z)}{1-z^2}\,dz\right)$