Review for the midterm exam (in class on Mon., March 4)

4. Suppose that f is an entire function such that

$$\lim_{|z|\to\infty} |f(z)| = \infty.$$

$\exists\ R_0 > 0$ such that $|f(z)| > 1$ when $|z| > R_0$

a) Explain why f can have at most finitely many zeroes.

b) Prove that f must be a non-constant polynomial.

Hint: The in-class proof of partial fractions. ← applied to $\frac{1}{f}$

All zeroes of f are in $\overline{D_{R_0}(0)}$ ← compact

An infinite subset of a compact set has a limit pt. in the set.

If had infinitely many zeroes in $\overline{D_{R_0}(0)}$, they'd have a limit pt. Identity thm $\Rightarrow f \equiv 0$. ↯.

Now $\frac{1}{f}$ has finitely many poles. Subtract off princ parts

$$\frac{1}{f} - \sum^{N} R_j \leftarrow \text{entire}$$

R_j p.p. at a_j

$$\underbrace{\left(\frac{1}{f} - R_j\right)}_{\text{removable sing at } a_j} - \underbrace{\sum_{k \neq j} R_k}_{\text{nice at } a_j}$$

$\longrightarrow 0$ as $|z| \to \infty$.

So it is bdd entire fcn.

Liouville's $\Rightarrow$ const., which must $= 0$.

$\frac{1}{f} = \sum^{N} R_j$ is rational. So f is rational.

f has no poles! So $f = $ poly.

2. Suppose that u is a real valued $\mathcal{C}^2$-smooth harmonic function on a domain Ω. Prove that u is either constant, or the set where the gradient of u vanishes has no limit points in Ω.

u harmonic $\Longrightarrow f = u_x - i u_y$ analytic

$\mathcal{C}^1$

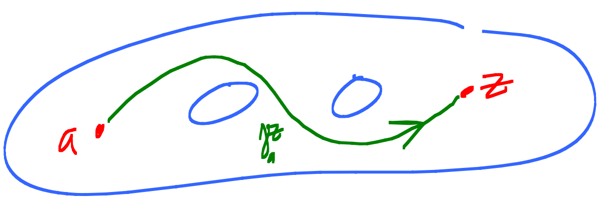
C-R Eqns ✓

$\{\nabla u = 0\}$ has lim pt $\Longrightarrow Z_f$ has a lim pt

$\Longrightarrow f \equiv 0$

$\Longrightarrow \nabla u \equiv 0$ on a domain

$\Longrightarrow u \equiv \text{const.}$



$$\int_{\gamma_a^z} \nabla u \cdot d\vec{s} = u(z) - u(a)$$

3. Suppose that f and g are analytic in a neighborhood of a. If f has a simple zero at a, then

$$\mathrm{Res}_a \frac{g}{f} = \frac{g(a)}{f'(a)}.$$

$$\mathrm{Res}_a \frac{g(z)}{z-a} = g(a)$$

Prove a similar formula in case f has a double zero at a, i.e., in case f is such that $f(a) = 0$, $f'(a) = 0$, but $f''(a) \neq 0$.

$$f(z) = a_2 (z-a)^2 + \cdots = (z-a)^2 \Big[\underbrace{a_2 + a_3(z-a) + \cdots}_{F(z)} \Big]$$

Note: $a_2 = \frac{f''(a)}{2!}$ $\quad a_3 = \frac{f'''(a)}{3!}$ $\quad \left(\text{and } F(a) = a_2 \quad \frac{F'(a)}{1!} = a_3 \right)$

Now $\displaystyle \frac{g(z)}{f(z)} = \frac{1}{(z-a)^2} \Big[\underbrace{\frac{g(z)}{F(z)}}_{A_0 + A_1(z-a) + A_2(z-a)^2 + \cdots} \Big]$ analytic around a

$$= \frac{A_0}{(z-a)^2} + \frac{A_1}{z-a} + \underbrace{A_2 + \cdots}_{\text{power series}}$$

Res_a

$$\mathrm{Res}_a \frac{g}{f} = A_1 = \frac{\frac{d}{dz}\left[\frac{g(z)}{F(z)}\right]}{1!}\Bigg|_{z=a} = \frac{g'(a)F(a) - F'(a)g(a)}{F(a)^2}$$

$$= \frac{g'(a)\frac{f''(a)}{2!} - \frac{f'''(a)}{3!}g(a)}{\left[\frac{f''(a)}{2!}\right]^2}$$

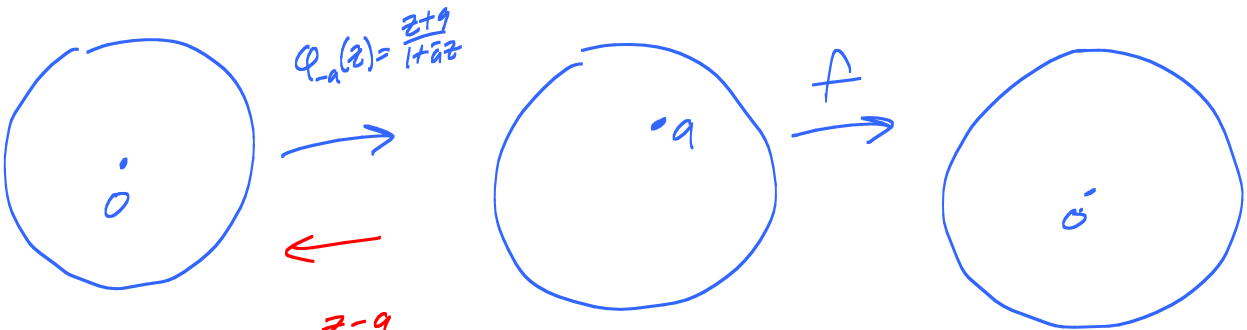
Hated formula:

$$\mathrm{Res}_a \frac{g}{f} = \frac{\lim\limits_{z \to a} \frac{d}{dz}\left[(z-a)^2 \frac{g(z)}{f(z)}\right]}{1!}$$

6. Show that if f is an analytic mapping of the unit disk into itself such that $f(a) = 0$, then

$$|f(z)| \le \left| \frac{z-a}{1-\bar{a}z} \right|$$

for all z in the disk.



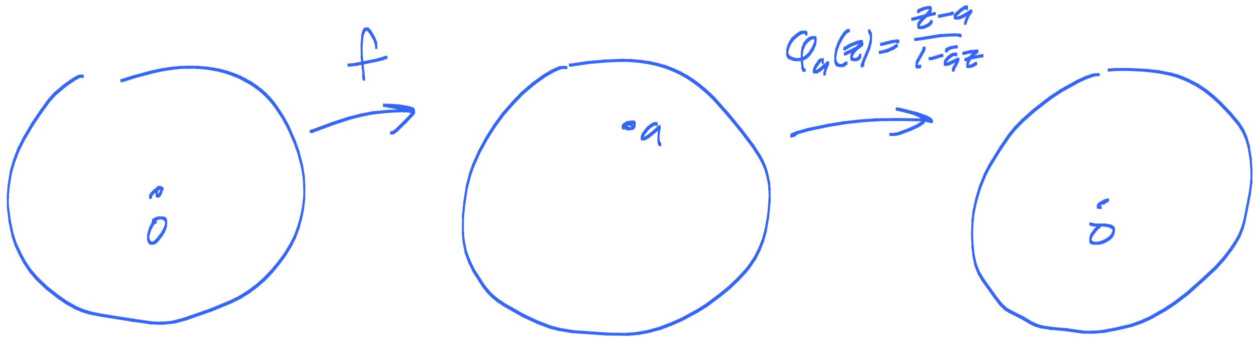
$$\varphi_a = \frac{z-a}{1-\bar{a}z}$$

$$\varphi_a^{-1} = \varphi_{-a} = \frac{z+a}{1+\bar{a}z}$$

$$\left| f\left(\underbrace{\frac{z+a}{1+\bar{a}z}}_{w} \right) \right| \le |z|$$

$$z = (\varphi_{-a})^{-1}(w) = \varphi_a(w) = \frac{w-a}{1-\bar{a}w} \quad ✓$$

7. Show that if f is an analytic mapping of the unit disk into itself, then $|f'(0)| \leq 1$.



$$F = \varphi_a \circ f$$

Schwarz: $\quad |F'(0)| \leq 1$

$$|\varphi_a'(\underbrace{f(0)}_{a})\, f'(0)| \leq 1 \qquad \varphi_a'(a) = \frac{1}{1-|a|^2}$$

So $\quad |f'(0)| \leq 1 - |a|^2 \leq 1$ ✓

What if $|f'(0)| = 1$? $\quad$ Then $|a|^2 = 0$.

So $a = 0$.

$f(0) = 0$! $\quad$ Schwarz part 2 $\Longrightarrow$

$f(z) = \lambda z$

where $|\lambda| = 1$.

10. Prove that if h_1 and h_2 are two analytic functions on a domain Ω such that $h_1^N \equiv h_2^N$ for some positive integer N, then there is an N-th root of unity λ such that $h_1 = \lambda h_2$ on Ω.

Case $h_1 \equiv 0$. Dumb.

Case $h_1 \not\equiv 0$: Then $\exists$ z_0 where $h_1(z_0) \neq 0$.

$$h_1(z_0)^N = h_2(z_0)^N$$

$$\left(\frac{h_1(z_0)}{h_2(z_0)}\right)^N = 1 \quad \text{so} \quad \frac{h_1(z_0)}{h_2(z_0)} = \lambda$$

an N-th root of unity.

Hmmm. Shrink a disc about z_0 so neither vanishes there.

Take a seq $z_n \to z_0$, $z_n \neq z_0$.

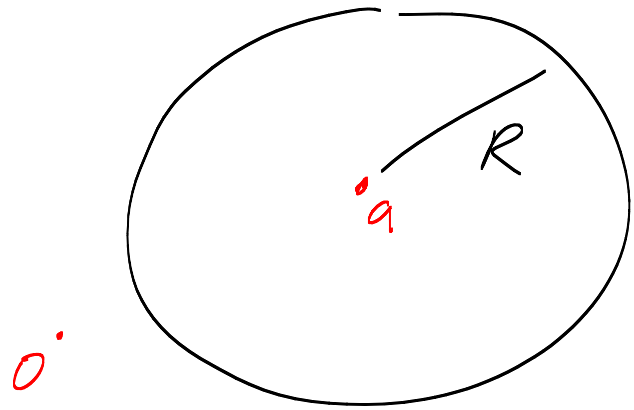
Get λ_n for each n.

Only N possible λ's. So one (say λ_0) gets repeated ∞ many times. Take subseq z_{n_k} where $\lambda_{n_k} = \lambda_0$.

$$\frac{h_1(z_{n_k})}{h_2(z_{n_k})} = \lambda_0 \qquad \underbrace{h_1 - \lambda_0 h_2}_{\equiv 0 \ \checkmark}$$

zeroes have lim pt

9. Suppose that f is an entire function that satisfies an estimate $|f(z)| \le C(1+|z|^N)$ for all z where C is a postivie constant and N is a positive integer. Prove that f must be a polynomial of degree N or less.



Cauchy Est $$|f^{(N+1)}(a)| \le \frac{(N+1)!\ \overbrace{\max_{C_R(a)} |f|}^{C(1+(|a|+R)^N)}}{R^{N+1}}$$

$\longrightarrow 0$ as $R \to \infty$.

a arbitrary. So $f^{(N+1)}(a) \equiv 0$.

$\Rightarrow$ f poly of deg N or less.

8. Suppose that f is an analytic function on the unit disc such that $|f(z)| < 1$ for $|z| < 1$. Prove that if f has a zero of order n at the origin, then $|f(z)| \le |z|^n$ for $|z| < 1$. How big can $|f^{(n)}(0)|$ be?

$$f(z) = z^n\left(a_n + a_{n+1} z + \cdots\right)$$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$$F(z) = \begin{cases} \dfrac{f(z)}{z^n} & z \neq 0 \\ \dfrac{f^{(n)}(0)}{n!} & z \neq 0 \end{cases}$$

Repeat proof of Schwarz.

Or Schwarz $\frac{f(z)}{z}$ $0 \to 0$ ✓

$\frac{f(z)}{z^2}$ $0 \to 0$ ✓

$\vdots$

$\frac{f(z)}{z^n}$ ✓

2. Prove that power series can be integrated term by term. To be precise, suppose that a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R > 0$ converges on the disc $D_R(0)$ to an analytic function $f(z)$. Prove that the power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$ also has radius of convergence R and that this series converges to an analytic anti-derivative of $f(z)$ inside the circle of convergence.

Diff term by

converges unif on compact subdiscs.

$$F(z) = \int_{L_0^z} f(w)\,dw = \sum_{n=0}^{\infty} a_n \int_{L_0^z} w^n\,dw$$