Lesson 10 Zeroes of analytic functions, the Identity theorem

Lemma: Suppose f is analytic on $D_R(z_0)$ and $f(z_0)=0$.

Then, either A) $f \equiv 0$ on $D_R(z_0)$, or

B) z_0 is an isolated zero of f, meaning $\exists\, \delta>0$ such that z_0 is the only zero of f in $D_\delta(z_0)$.

Pf: $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ Case A: $\forall\, a_n = 0$.

Case B: a_N first $a_n \neq 0$. $f(z_0)=0, \dots, f^{(N-1)}(z_0) = 0$

$$f(z) = a_N (z-z_0)^N + \cdots = (z-z_0)^N \underbrace{\left(a_N + a_{N-1}(z-z_0) + \cdots\right)}_{H(z),\ \text{same R.of C. as series for } f!}$$

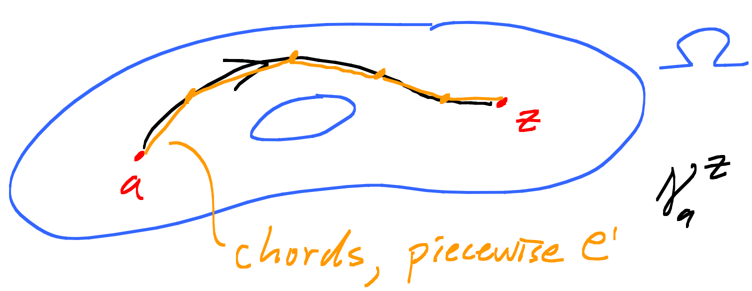
$$= (z-z_0)^N \underbrace{H(z)}_{H(z_0) = a_N \neq 0}$$

Def^n: An open connected set in $\mathbb{C}$ is called a domain.

Def^n #1: If Ω is open, we say it is path connected if any two pts in Ω can be connected by a piecewise $\mathcal{C}^1$-smooth curve in Ω.

Remark: Topology MA 571 uses continuous curves in def^n.

Gives same concept:



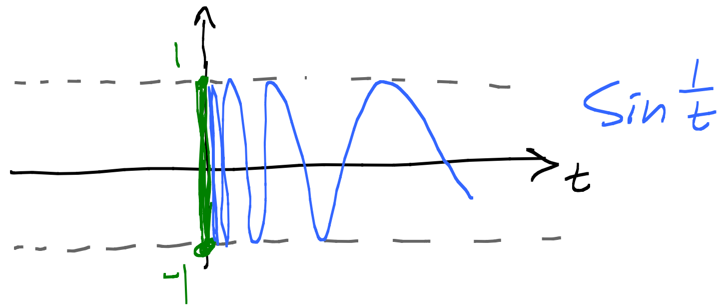
Def^n #2: Ω open is connected if, whenever

$$\Omega = U_1 \cup U_2 \quad \text{where } U_1, U_2 \text{ open}$$
$$\text{and } U_1 \cap U_2 = \emptyset,$$

one U_j must be empty and the other $= \Omega$.

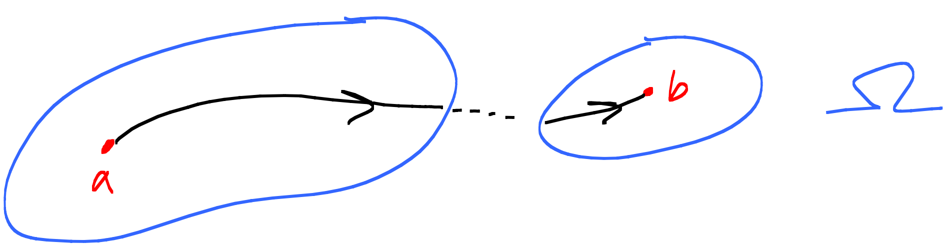
Fact: Def #1 and #2 are equiv for open sets in $\mathbb{C}$.

EX:



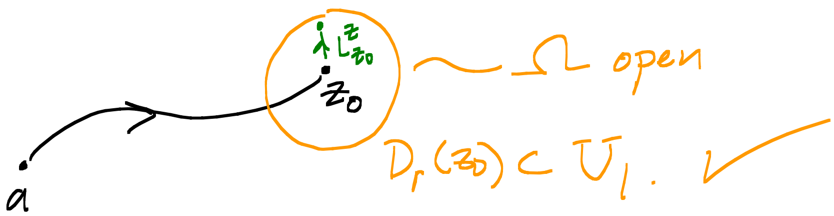
S = segment $\cup$ graph
Connected,
but not
path connected

Pf of Fact: 1) Ω open connected #2.



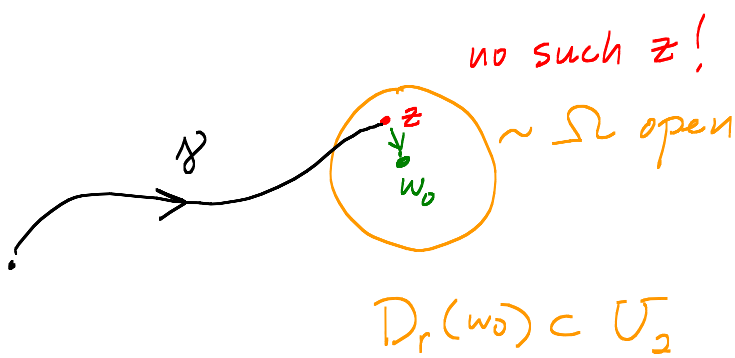
Let $a \in \Omega$. Let $U_1 = \{z : \exists\ \gamma_a^z \subset \Omega\}$

U_1 is open:



$U_2 = \Omega - U_1$.

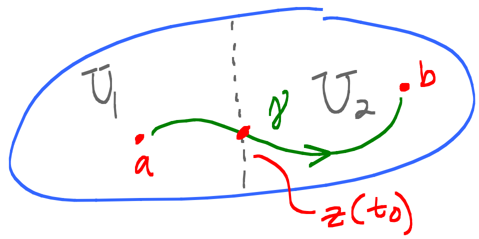
U_2 is open. $w_0 \in U_2$:



Conclude $U_2 = \emptyset$ (because $a \in U_1$).

2) Suppose Ω is path-connected, and $\Omega = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$.

Suppose $a \in U_1$, $b \in U_2$.



$\gamma: z(t) : \alpha \le t \le \beta$

Let t_0 be

$\sup\{t : z(\tau) \in U_1, \tau \in [\alpha, t)\}$

Where is $z(t_0)$? In Ω, but not U_1 or U_2! ↯.

MA 530: Analytic fcns on domains.

Thm: If f is analytic on a domain Ω and $f' \equiv 0$, then f is const.

Pf: Fix $z_0 \in \Omega$. For $z \in \Omega$, let $\gamma_{z_0}^{z}$ be a $\mathcal{C}^1$-path in Ω connecting z_0 to z.

$$0 = \int_{\gamma_{z_0}^{z}} f' \, dw = f(z) - f(z_0). \quad \checkmark$$

Theorem: (Identity Thm) Suppose f is analytic on a domain Ω. Let $Z_f = \{z \in \Omega : f(z) = 0\}$.

Then either A) $f \equiv 0$ on Ω, or

B) Z_f has no limit points in Ω.

Def^n: z_0 is a limit pt of Z_f means $\exists$ seq $z_n \in Z_f$ with $z_n \neq z_0$ for all n and $\lim_{n\to\infty} z_n = z_0$.

Note: If $z_0 \in \Omega$ and z_0 is a limit pt of Z_f, then

$$f(z_0) = \lim_{n\to\infty} \underbrace{f(z_n)}_{0} = 0 \quad \text{by continuity.}$$

Pf of Thm: Suppose Z_f has a limit pt in Ω.

Let U_1 = set of all limit pts of Z_f in Ω.

Claim: U_1 is open: If $z_0 \in U_1$, then Ω open $\Rightarrow$ $\exists$ $D_r(z_0) \subset \Omega$. z_0 is not an isolated zero of f. Lemma $\Rightarrow$ $f \equiv 0$ on $D_r(z_0)$. So $D_r(z_0) \subset U_1$.

Next, show $U_2 = \Omega - U_1$ is open. Suppose $w_0 \in U_2$.

Case 1: $f(w_0) \neq 0$. Then $\exists$ $\delta > 0$ such f is nonvanishing on $D_\delta(w_0) \subset \Omega$. See $D_r(w_0) \subset U_2$.

Case 2: $f(w_0) = 0$. w_0 not a limit pt of zeroes $\Rightarrow$ w_0 is an isolated zero. f non-vanishing on $D_\delta(w_0) - \{w_0\}$. See $D_\delta(w_0) \subset U_2$.

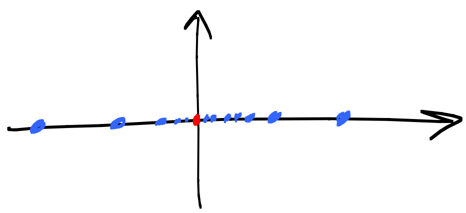
Done: $U_1 \neq \emptyset$. So $U_2 = \emptyset$ and $U_1 = \Omega$.

$U_1 \subset Z_f$. So $Z_f = \Omega$.

Cor: If two analytic fcns f, g a domain Ω agree on a set with a limit pt in Ω, the $f \equiv g$ on Ω.

b) If f and g agree on $D_\varepsilon(z_0) \subset \Omega$, then $f \equiv g$ on Ω.

EX: $\sin \frac{1}{z}$ has zeroes at $z = \frac{1}{n\pi}$, $n \in \mathbb{Z}$.



zeroes pile up at origin.

OK. $\Omega = \mathbb{C} - \{0\}$. OK for zeroes to pile up at a boundary pt.

Identity Theorem way false $\mathbb{R} \to \mathbb{R}$:

$$h(t) = \begin{cases} e^{-1/t^2} \sin \frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

h is C^∞ smooth on $\mathbb{R}$, zeroes pile up at $t=0$.

Taylor series at origin $\equiv 0 \neq h$ near zero.

Identity Thm $\Rightarrow$ e^z is only way to extend e^x from $\mathbb{R}$ to $\mathbb{C}$ as a complex diff'ble fcn!