Lesson 11 The Maximum principle

Hwk 3: Know $\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta) \leftarrow$ true for $\theta \in \mathbb{C}$ $\qquad \xrightarrow{t_0} \mathbb{R}$

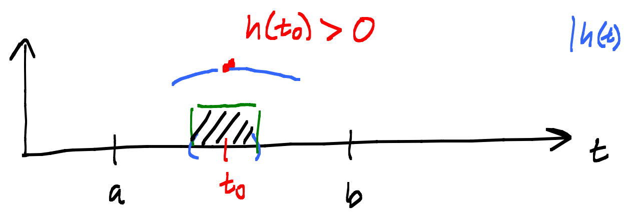
$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \cos\beta\sin\alpha \leftarrow$ true for $\alpha, \beta \in \mathbb{C}$

<u>Calculus lemma</u>: $h: [a,b] \to \mathbb{R}$ continuous and $h(t) \geq 0$ on $[a,b]$.

Then $\int_a^b h(t)\,dt = 0 \implies h \equiv 0$.

$|h(t) - h(t_0)| < \varepsilon = \frac{h(t_0)}{2}$ if $|t - t_0| < \delta$

<u>Pf</u>:

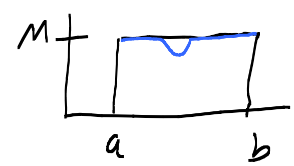


See $\int h > 2\delta \cdot \varepsilon > 0$ $\lightning$.

<u>Lemma</u>: $h: [a,b] \to \mathbb{R}$, $h(t) \leq M$ on $[a,b]$, <u>continuous</u>.

$$\int_a^b h(t)\,dt = M(b-a) \implies h(t) \equiv M \text{ on } [a,b].$$

<u>Pf</u>: Use h to be $M-h$ in lemma above.



Cauchy integral formula $\implies$ <u>Averaging property</u> for analytic fcns.

$$f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{(z_0 + re^{it}) - z_0}\, ire^{it}\,dt \qquad C_r(z_0): \begin{array}{l} z(t) = z_0 + re^{it} \\ z'(t) = ire^{it} \end{array}$$

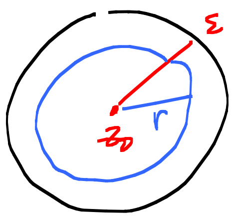
$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt = \frac{1}{2\pi r}\int_0^{2\pi} f(z_0 + re^{it})\,\underbrace{r\,dt}_{ds}$$

$$= \text{Average of } f \text{ on } C_r(z_0).$$

Averaging prop $\Rightarrow$ Max princ.

Maximum modulus principle #1: Suppose f analytic on a domain Ω. If $|f|$ attains a local max at a point $z_0 \in \Omega$ (meaning $\exists\ \varepsilon > 0$ such that $D_\varepsilon(z_0) \subset \Omega$ and $|f(z)| \le |f(z_0)|$ for all $z \in D_\varepsilon(z_0)$), then $f \equiv$ const on Ω.

Pf: Suppose $|f|$ has a local max at z_0. Get $D_\varepsilon(z_0) \subset \Omega$ as above.



$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt$$

$|f(z_0)| = M$

$$\underbrace{|f(z_0)|}_{=M} \le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0+re^{it})|}_{\le M}\,dt \le \frac{1}{2\pi} M \cdot 2\pi$$

Only way to get equality is if $|f(z_0+re^{it})| \equiv M$, $t \in [0, 2\pi]$.

True for $0 < r < \varepsilon$. So $|f| \equiv M$ on $D_\varepsilon(z_0)$.

Lemma: $|f| \equiv M$ on $D_\varepsilon(z_0) \Rightarrow f \equiv$ const there.

Pf of lemma: $f(x+iy) = u(x,y) + iv(x,y)$. $\quad |f|^2 = M^2$

(*) $\quad u^2 + v^2 = M^2$

Case $M=0$. $f\equiv 0$. ✓

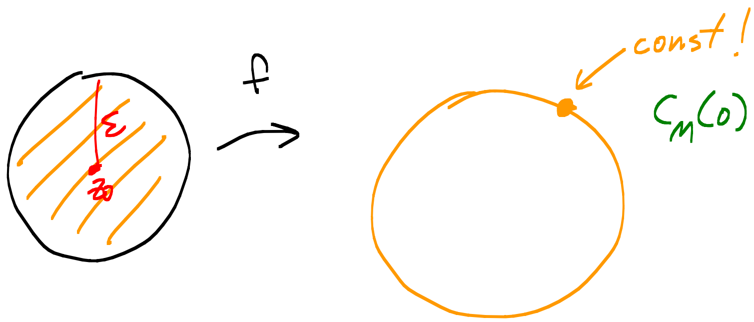
Case $M\neq 0$. $\begin{cases} \frac{\partial}{\partial x}(*) : & 2u\,u_x + 2v\,v_x = 0 \\ \frac{\partial}{\partial y}(*) : & 2u\,\overset{=-v_x}{u_y} + 2v\,\overset{=u_x}{v_y} = 0 \end{cases}$

$$\underbrace{\begin{bmatrix} u_x & v_x \\ -v_x & u_x \end{bmatrix}}_{\text{det must}=0!} \begin{pmatrix} u \\ v \end{pmatrix} = \vec{0}$$

↑ not the zero vector!

$$\det = (u_x)^2 + (v_x)^2 = |\underbrace{u_x + i v_x}_{\text{red box \#1}}|^2 = |f'|^2 \equiv 0$$

So $f\equiv$ const. HWK 3: $\rho = |z|^2 - M^2$



Pf: back to max princ pf: Have $f\equiv$ const on $D_\varepsilon(z_0)\subset\Omega$.

Identity Thm $\Rightarrow$ $f\equiv$ const on Ω.

Max princ #2: Suppose Ω is a bounded domain ($\Omega\subset D_R(0)$ for some $R>0$) and f is continuous on $\overline{\Omega}$ (the closure of Ω in $\mathbb{C}$) and f is analytic on Ω. Then $|f|$ attains its max modulus at a pt on the boundary. ($\exists\, z_0$ in bndry with $|f(z_0)| = \underset{\overline{\Omega}}{\text{Max}} |f|$.)

<u>Pf</u>: f continuous $\Rightarrow$ $|f|$ continuous.

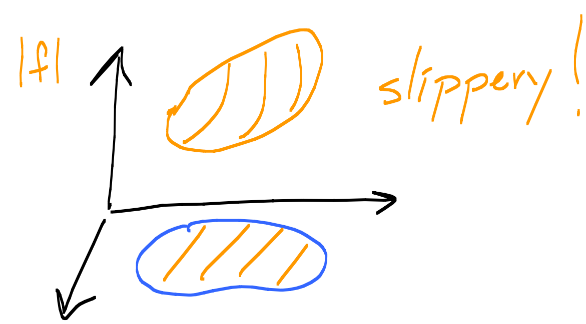
A continuous fcn on a compact set (closed and bounded set) <u>attains</u> its max at a pt $z_0 \in \overline{\Omega}$

<u>Case 1</u>: $z_0 \in b\Omega = \overline{\Omega} - \Omega$. ✓

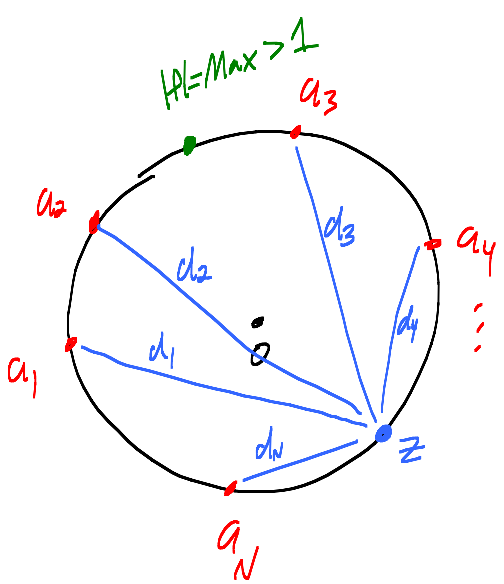
<u>Case 2</u>: $z_0 \in \Omega$. Max princ #1 $\Rightarrow$ f const.

So $|f| = M$ on Ω. Every pt in $\overline{\Omega} - \Omega$ is a limit pt of Ω. Continuity $\Rightarrow$ $|f| = M$ on $\overline{\Omega}$.

So max is <u>attained</u> on bndry.



<u>Problem</u>:



Unit disc

Show $\exists$ z with $|z| = 1$ such that $\prod_{n=1}^{N} d_n = 1$.

Let $f(z) = \prod_{n=1}^{N} (z - a_n)$. Notice $|f(z)| = \prod_{1}^{N} d_n$ when $|z| = 1$.

f is poly : $f(z) = z^N + \cdots$ <u>not</u> constant.

Aha! $|f(0)| = \left|\prod_1^N (-a_n)\right| = \prod_1^N |a_n| = 1$.

Aha! 1 can't be max because $f \not\equiv$ const.

So Max is > 1. M.P. #2 says M is <u>attained</u> on $|z|=1$,

$|f| = 0$ at a_n's. Indermediate value thm $\Rightarrow \exists$ pt between a_1 and place where max modulus $M > 1$ is attained where $|f| = 1$. ✓

<u>Min princ?</u> z^n shows <u>no</u>, but if $f(z)$ analytic and <u>non-vanishing</u>, then max princ applied to $\frac{1}{f(z)}$ gives the min. princ.

<u>EX</u>: f analytic on $D_1(0)$, continuous on $\overline{D_1(0)}$.

If $f: C_1(0) \to C_1(0)$ and f non-vanishing on $D_1(0)$, then $f \equiv$ const. of modulus 1.