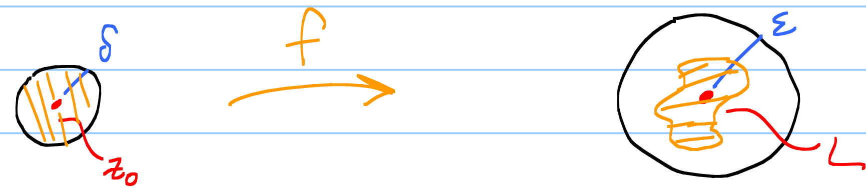
Lecture 2

HWK 1 due Thurs, Jan 20 at 11:59 pm in Gradescope

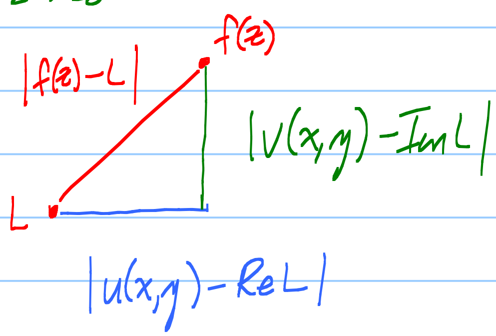
$\lim_{z \to z_0} f(z) = L$: Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|f(z) - L| < \varepsilon$ when $|z - z_0| < \delta$, $z \neq z_0$.



Little lemma : $f(x+iy) = u(x,y) + i\,v(x,y)$ $\qquad z_0 = x_0 + i y_0$

$$\lim_{z \to z_0} f(z) = L \iff \begin{cases} \lim_{(x,y)\to(x_0,y_0)} u(x,y) = \operatorname{Re} L \\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) = \operatorname{Im} L \end{cases}$$

Pf:



$\Rightarrow$ $|\text{Leg}| \le |\text{Hyp}|$

$\Leftarrow$ Pyth

Def^n: f is $\mathbb{C}$-diff'ble at a :

$$\lim_{z \to a} \underbrace{\frac{f(z) - f(a)}{z - a}}_{DQ} \text{ exists. Call limit } f'(a).$$

EX: $f(z) = z^2$ $\qquad DQ = \dfrac{z^2 - a^2}{z - a} = z + a \longrightarrow 2a$ as $z \to a$.

$f'(a) = 2a$. $\qquad f'(z) = 2z$.

Fact: Basic rules of limits and differentiability $\mathbb{R} \to \mathbb{R}$ go over word for word $\mathbb{C} \to \mathbb{C}$. Proofs too!

EX: $[fg]' = f'g + fg'$ etc. Chain rule too.

EX: $f(z) = \bar{z}$ $\quad f(x+iy) = x - iy$ is <u>not</u> $\mathbb{C}$-diff'ble!

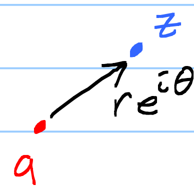
<u>Facts</u>: $|zw| = |z||w|$ $\quad \overline{zw} = \bar{z} \cdot \bar{w}$

$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$, $w \neq 0$ $\quad \overline{\left(\frac{z}{w}\right)} = \bar{z}/\bar{w}$, $w \neq 0$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta = \overline{e^{i\theta}} = \frac{1}{e^{i\theta}} \qquad (N=-1, \text{ De Moivre})$$

$$DQ = \frac{\bar{z} - \bar{a}}{z - a}$$



$z = a + re^{i\theta}$

$$= \frac{\overline{(re^{i\theta})}}{re^{i\theta}} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-i\theta} \cdot e^{-i\theta} = e^{-2i\theta}$$

Ouch! Limit as I slide z into a along a ray is diff for diff directions! So complex limit DNE.

<u>Def^n</u>: (Open set Ω) f is <u>analytic</u> on Ω if it is $\mathbb{C}$-diff at every pt in Ω. (<u>holomorphic</u>)

<u>Def^n</u>: Continuous: $\lim_{z \to z_0} f(z) = f(z_0)$

<u>Thm</u>: $\mathbb{C}$-diff'ble at $a \Rightarrow$ continuity at a.

<u>Pf</u>: $$\frac{f(z) - f(a)}{z - a} = f'(a) + E(z) \quad \leftarrow \text{defines } E(z)$$

$\mathbb{C}$-diff'ble $\Rightarrow$ $E(z) \to 0$ as $z \to a$.

Solve for $f(z)$:

$$f(z) = f(a) + f'(a)(z-a) + E(z)\cdot(z-a)$$

$z \to a$

$$f(a) + f'(a)\cdot 0 + 0\cdot 0 = f(a) \checkmark$$

$$f(z) = (\mathbb{C}\text{-linear thing}) + (\text{Really small})$$

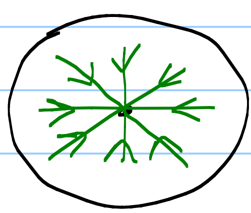
Cauchy-Riemann equations (CR-eqns)

If $f(x+iy) = u(x,y) + i v(x,y)$ is $\mathbb{C}$-diff'ble at $z_0 = x_0 + iy_0$, then the first partials of u and v exist and

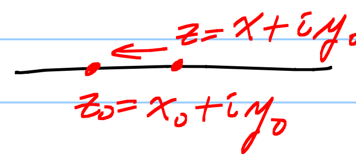
CR-eqns
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad \text{at } (x_0, y_0)$$

Notation: $u_x = \dfrac{\partial u}{\partial x}$ $\quad u^0 = u(x_0, y_0)$ $\quad u_x^0 = \dfrac{\partial u}{\partial x}(x_0, y_0)$, etc

Pf:



Step 1: Let $z \to z_0$ along horiz



$$DQ = \frac{[u(x,y_0) + i v(x,y_0)] - [u^0 + i v^0]}{(x+iy_0) - (x_0 + iy_0)}$$

$$= \left(\frac{u(x,y_0) - u^0}{x - x_0}\right) + i\left(\frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}\right)$$

Little lem

Re $f'(z_0)$ Im $f'(z_0)$

So first partials exist and

$$\boxed{f'(z_0) = u_x(x_0,y_0) + i\,v_x(x_0,y_0)}$$

Red box #1

Step 2 Vertical

$z = x_0 + iy$

$z = x_0 + iy_0$

$$DQ = \frac{\text{eeel}}{(x_0+iy)-(x_0+iy_0)} = \frac{1}{i}\cdot\frac{\text{eeel}}{(y-y_0)}$$

$$= \underbrace{\frac{1}{\underset{-i}{i}}\;\frac{u(x_0,y)-u(x_0,y_0)}{y-y_0}} + \frac{v(x_0,y)-v(x_0,y_0)}{y-y_0}$$

Re $f'(z_0)$ + i Im $f'(z_0)$

So u_y, v_y exist and

$$\boxed{f'(z_0) = v_y(x_0,y_0) - i\,u_y(x_0,y_0)}$$

Red box #2

Red box #1 = #2. C-R eqns ✓

EX: $\bar{z} = x - iy$. One of C-R eqns fail.

EX: $E(x+iy) = \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$ $\quad (e^z)$

C-R eqns hold! Is $E(z)$ analytic? Yes, because

Partial converse to C-R eqns: Suppose u, v are $\mathcal{C}^1$-smooth (meaning that u, v, and all first partials are cont.) on Ω, open and they satisfy the C-R eqns. Then $f = u + iv$ is analytic on Ω.

Looman-Menchoff thm:
1) u, v continuous on Ω
2) first partials exist and
3) satisfy the C-R eqns.

Then $u + iv$ is analytic on Ω.

Narasimhan: Complex analysis in one variable, p. 43-50.

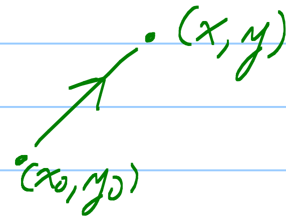
Calculus fact: Suppose u is $\mathcal{C}^1$-smooth.

$$u(x,y) = u^0 + u_x^0 \cdot (x - x_0) + u_y^0 \cdot (y - y_0) + R_u(x,y)$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{R_u(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

Pf: (A): $u(x,y) - u^0 = \int_0^1 \frac{\partial}{\partial t}\left[u\big(x_0 + t(x-x_0),\ y_0 + t(y-y_0)\big)\right] dt$



$$= \int_0^1 u_x(\quad)\cdot(x-x_0) + u_y(\quad)\cdot(y-y_0)\ dt$$

(B): $u_x^0\cdot(x-x_0) + u_y^0\cdot(y-y_0)$

$$= \int_0^1 u_x^0\cdot(x-x_0) + u_y^0\cdot(y-y_0)\ dt$$

$$R(x,y) = (A) - (B) = \int_0^1 \underbrace{\left[u_x(\ ) - u_x^0\right]}_{\varepsilon}\cdot(x-x_0) + \underbrace{\left[u_y(\ ) - u_y^0\right]}_{\varepsilon}(y-y_0)\,dt$$

$$\frac{|R(x,y)|}{|z-z_0|} \le \int_0^1 \varepsilon\,\frac{|x-x_0|}{|z-z_0|} + \varepsilon\,\frac{|y-y_0|}{|z-z_0|}\ dt$$

$$\le 2\varepsilon \quad ✓$$