Lecture 4 Complex power series

HWK 1 due Thurs 11:59 pm in GS
HWK 2 due the next Thurs.

How to cook up harmonic conjugate v for harmonic C^2 harmonic fcn u:

Want v so that $u+iv$ is analytic: C-R equns $u_x = v_y$, $u_y = -v_x$ $\to$ $v_x = -u_y$, $\underbrace{v_y = u_x}_{\nabla v = \vec{F}}$

where $\vec{F} = (-u_y, u_x)$

Prob: Find v with $\nabla v = \vec{F}$

potential fcn for $\vec{F}$

$\nabla v = \underbrace{\vec{F}}_{(v_x,\, v_y)}$

in $\mathbb{R}^3$: $\iff$ Curl $\vec{F} \equiv 0$

$\underbrace{\text{in } \mathbb{R}^2}_{\text{or } \Omega \text{ with no holes}}$: $\iff$ $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$ $\leftarrow$ $(v_{yx} = v_{xy})$

Shown in MA 261. Check: $\dfrac{\partial}{\partial y}\big(\underbrace{-u_y}_{F_1}\big) \overset{?}{=} \dfrac{\partial}{\partial x}\big(\underbrace{u_x}_{F_2}\big)$

Yes! $u_{yy} + u_{xx} \equiv 0$ ✓

Fact: v is unique up to $+C$.

Power series:

Lem: $\sum_1^\infty a_n$ conv $\implies$ $a_n \to 0$.

Pf: $S_N = \sum_1^N a_n$ $\qquad a_N = S_N - S_{N-1} \to L - L$ as $N \to \infty$ ✓

Def^n: $\sum_1^\infty a_n$ conv absolutely if $\sum_1^\infty |a_n| < \infty$.

Lemma: abs conv $\implies$ conv.

Pf: Abs conv $\implies$ S_N Cauchy seq $\implies$ $\sum_1^\infty a_n$ conv.

Big Fact: Power series have a radius of conv. R, meaning

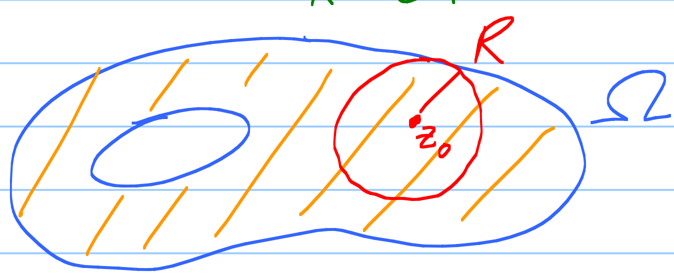
1) $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ conv. abs. in $D_R(z_0)$

2) conv uniformly on $D_r(z_0)$, $r < R$.

3) div if $|z-z_0| > R$.

$R=0$ case: only conv at z_0.
$R=\infty$ case: conv for all $z \in \mathbb{C}$.

Big Fact: Analytic fcns are given locally by conv power series with $R>0$.



Pf: later

Hadamard's Formula: $$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Def^n: $\limsup_{n\to\infty} \overset{\in\mathbb{R}}{r_n} = \lim_{N\to\infty} \underbrace{\sup\{r_n : n \geq N\}}_{\text{non-increasing in } N}$

So limit exists as $N\to\infty$.
(Might be $-\infty$.)
(But, if $r_n = \sqrt[n]{|a_n|}$, $\limsup \geq 0$.)

Later: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Hadamard's: $\frac{1}{R} = \limsup_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$

Stirling's formula:

$$\underbrace{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n!}\ < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{4n}\right)$$

$$\frac{1}{\sqrt[n]{n!}} < \frac{1}{(2\pi n)^{1/2n}\left(\frac{n}{e}\right)} \longrightarrow 0 \quad \text{as } n\to\infty.$$

So $\frac{1}{R}$ "$=$" 0 and $R=\infty$.

Ratio test: much easier!

Geometric series: $\sum_{n=0}^{\infty} z^n$

$|z|\geq 1$ terms $\not\to 0$. So div.

$$S_N = 1 + z + \cdots + z^N$$
$$zS_N = \quad z + \cdots + z^N + z^{N+1}$$
$$(1-z)S_N = 1 - z^{N+1}$$

$$S_N = \frac{1}{1-z} + \underbrace{\frac{-z^{N+1}}{1-z}}_{E_N(z)}$$

Δ ineq:

$|z+w| \leq |z| + |w|$ $\leftarrow$ Numerator estimate

$|z-w| \geq \big||z|-|w|\big|$ $\leftarrow$ Denom est

$|z+w| \geq \big||z|-|w|\big|$ (replace w by $-w$)

If $|z| < r < 1$, then

$$|1-z| \geq \big|1-|z|\big| = 1-|z| > 1-r.$$

So $$\left|E_N(z)\right| = \frac{|z|^{N+1}}{|1-z|} \leq \underbrace{\frac{|z|^{N+1}}{1-|z|}}_{\text{see } \sum z^n \text{ conv}} < \underbrace{\frac{r^{N+1}}{1-r}}_{\text{see unif conv.}} \to 0 \text{ as } N\to\infty$$

<u>Note</u>: $E_N(x) = \frac{-x^{N+1}}{1-x} \longrightarrow -\infty$ as $x \nearrow 1$.

Do <u>not</u> get unif conv on $D_1(0)$.

<u>Pf of Hadamard's.</u> Case $0 < R < \infty$. Assume $z_0 = 0$.

Pick ρ, r with $|z| < r < \rho < R$ = Hadamard's

$$\frac{1}{R} < \frac{1}{\rho}$$

$$\sup_{n \geq N} \sqrt[n]{|a_n|}$$

sliding down to $\frac{1}{R}$

So, $\exists\ N$ such that $\sqrt[n]{|a_n|} < \frac{1}{\rho}$ when $n \geq N$.

$$|a_n| < \frac{1}{\rho^n} \quad \text{when } n \geq N$$

$$|a_n z^n| < \left(\frac{|z|}{\rho}\right)^n \quad \text{when } n \geq N.$$

< 1

Aha! Comparison with geom series $\sum \left(\frac{|z|}{\rho}\right)^n$ shows $\sum_0^\infty a_n z^n$ is abs. conv., Can make r as close to R as desired.

So $\sum a_n z^n$ conv abs in $D_R(0)$.

Next: Get unif conv on $D_r(0)$. $f(z) = \sum_{n=0}^\infty a_n z^n$

$$\left| f(z) - \sum_{n=0}^M a_n z^n \right| = \left| \sum_{n=M+1}^\infty a_n z^n \right|$$

$$\leq \sum_{n=M+1}^{\infty} |a_n z^n| \quad \text{and if } M>N:$$

$$\leq \sum_{n=M+1}^{\infty} \left(\frac{|z|}{\rho}\right)^n = \frac{\left(\frac{|z|}{\rho}\right)^{M+1}}{1-\left(\frac{|z|}{\rho}\right)}$$

$$< \frac{\left(\frac{r}{\rho}\right)^{M+1}}{1-\frac{r}{\rho}}$$

$$\to 0 \text{ as } M \to \infty.$$

Get unif conv on $D_r(0)$.

Last step: Div if $|z|>R$.

$$\frac{1}{|z|} < \frac{1}{R} \leq \sup_{n \geq N} \sqrt[n]{|a_n|}$$

So, for each N, $\exists\, n$ with $n \geq N$ such that

$$\frac{1}{|z|} < \frac{1}{R} \leq \sqrt[n]{|a_n|}$$

$$1 \leq \sqrt[n]{|a_n|}\, |z| \quad \leftarrow \text{raise to } n\text{-th power}$$

$$1 \leq |a_n z^n|$$

Aha! This shows that terms $a_n z^n \not\to 0$.
So series diverges.
$R=0, \infty$ cases are similar and easier.

EX: $\sum_0^{\infty} n!\, z^n$. $R=\infty$

EX: $\sum_{0}^{\infty} \frac{z^n}{n^2}$, $R=1$ Converges uniformly on $\underbrace{\overline{D_1(0)}}_{}$

$\{z : |z| \leq 1\}$
closed unit disc.