Lecture 5 Complex integration

HWK 2 due in GS
Thurs, Jan. 27, 11:59 pm

Ratio test Suppose $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$ exists $= L$

$L<1$: $\sum_1^\infty z_n$ converges. $\leftarrow$ compare tail end of series with convergent geom series

$L>1$: $\sum_1^\infty z_n$ diverges $\leftarrow$ terms $\not\to 0$

$L=1$: test fails : e.g. $\sum \frac{1}{n^2}$ conv, $\sum \frac{1}{n}$ div

Remark: Suppose $\sum_0^\infty a_n z^n$ has R.of C. R. Since $\lim_{n\to\infty} \sqrt[n]{n} = 1$,

Hadamard's $\Rightarrow$ $\sum_{n=1}^\infty n a_n z^{n-1}$ has same R. of C. R.

Big fact: Power series converge to analytic fcns inside $D_R(z_0)$,

$R =$ R.of C., and can diff'tiated term by term,

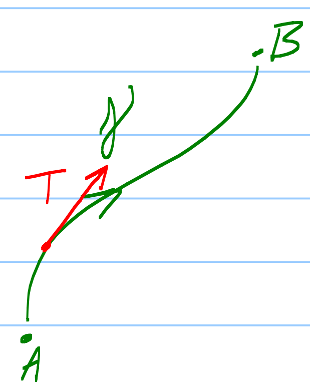
R. of C. of f' = R. of C. of f.

Stein p. 16–18.

Bigger fact: Analytic fcns are given locally by conv. power series.

Pf's: Later, sneaky.

Integration along paths



γ: $z(t) = x(t) + i y(t)$

$a \le t \le b$

$z : [a,b] \to \mathbb{C}$

Def^n: $tr(\gamma)$ = "trace of γ" $= \{z(t) : t \in [a,b]\}$

$$z'(t_0) = \lim_{t \to t_0} \frac{z(t) - z(t_0)}{t - t_0} = \lim (DQ\text{-}x) + i\,(DQ\text{-}y)$$

$$= x'(t_0) + i\,y'(t_0)$$

Complex tangent vector $T = z'(t)$

Unit complex tang vect $\hat{T} = z'(t) / |z'(t)|$

Chain rule #1

$g : \Omega_1 \to \Omega_2$ analytic

$f : \Omega_2 \to \mathbb{C}$ analytic

$h(z) = f(g(z))$ is analytic and $h'(z) = f'(g(z))\,g'(z)$

Pf: Hwk 1 or Baby Rudin.

Chain rule #2: f analytic on an open set containing $tr(\gamma)$ and $z(t)$ is diff^{l}ble, then

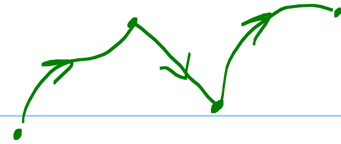
$$\frac{d}{dt} f(z(t)) = f'(z(t))\,z'(t)$$

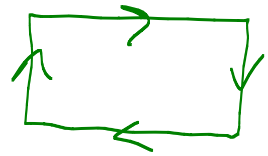
Pf: $f(z(t)) = u(x(t), y(t)) + i\,v(x(t), y(t))$

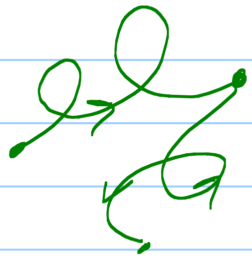
$$\frac{d}{dt} f(z(t)) = \left(u_x \dot{x} + \overset{= -v_x}{u_y} \dot{y}\right) + i\left(v_x \dot{x} + \overset{= u_x}{v_y} \dot{y}\right)$$

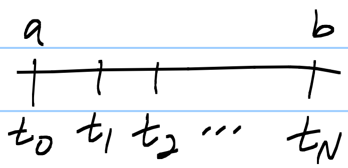
$$= \underbrace{[u_x + i v_x]}_{\text{red box \#1}} \cdot (\dot{x} + i \dot{y}) \checkmark$$

Piecewise $\mathcal{C}^1$ paths









$z(t)$ cont on $[a,b]$,
$z(t)$ continuously diff'ble on (t_n, t_{n+1}),
$z'(t)$ extends continuously to $[t_n, t_{n+1}]$

<u>Def^n</u>: $$\int_a^b z(t)\,dt = \lim_{\Delta t \to 0} \sum_{n=1}^{N} z(t_n)\Delta t_n$$

$$= \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$$

<u>EX</u>: $$\int_0^{2\pi} e^{it}\,dt = \int_0^{2\pi} \cos t\,dt + i\int_0^{2\pi} \sin t\,dt = 0$$

<u>Fund Thm Calc #1</u>: $$\int_a^b z'(t)\,dt = z(b) - z(a)$$

when $z(t)$ is continuously diff'ble.

True for piecewise $\mathcal{C}^1$ $z(t)$ too:

$$\int_a^b z'(t)\,dt = \sum_{n=0}^{N-1} \underbrace{\int_{t_n}^{t_{n+1}} z'(t)\,dt}_{z(t_{n+1}) - z(t_n)} = z(t_N) - z(t_0)$$

"telescopes"

<u>Pf</u> real and imag parts. Freshman calc.

Lemma: $\left| \int_a^b z(t)\, dt \right| \le \int_a^b |z(t)|\, dt$

Think: $\left| \sum z(t_n) \Delta t_n \right| \le \sum |z(t_n)| \Delta t_n$

Let $\Delta t \to 0$.

Pf: $\int_a^b z(t)\, dt = re^{i\theta}$ $\qquad \lambda = e^{-i\theta}$

$$\left| \int_a^b z(t)\, dt \right| = r = \lambda \int_a^b z(t)\, dt$$

$$\underset{\substack{\uparrow \\ \text{check}}}{=} \int_a^b \lambda z(t)\, dt$$

$$= \int_a^b \underbrace{\mathrm{Re}\, \lambda z(t)}_{\le \underbrace{|\lambda z(t)|}_{\underbrace{|\lambda|}_{1} |z(t)|}}\, dt + i \underbrace{\int_a^b \mathrm{Im}\, \lambda z(t)\, dt}_{\text{must be zero}}$$

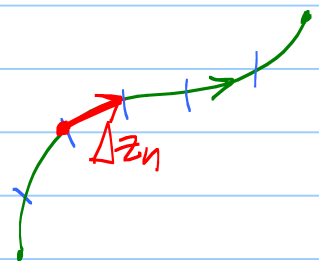
$$\le \int_a^b |z(t)|\, dt \quad \checkmark$$

Def^n: Suppose $f: tr(\gamma) \to \mathbb{C}$ is continuous.

$$\int_\gamma f\, dz \underset{\substack{\uparrow \\ \text{def}^n}}{=} \int_a^b f(z(t)) \underbrace{z'(t)\, dt}_{dz}$$

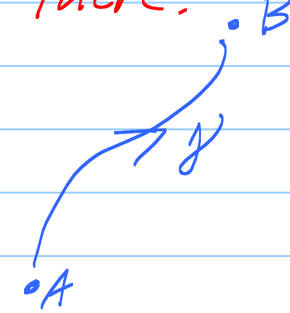
Konrad Knopf

$$\int_\gamma f\,dz = \lim_{\Delta t \to 0} \sum f(z(t_n))\, \underbrace{\Delta z_n}_{\substack{z(t_{n+1}) - z(t_n) \\ \approx z'(t_n)(t_{n+1}-t_n)}}$$



<u>Fund Thm Calc #2</u>: f analytic on an open set containing $tr(\gamma)$ <u>and</u> f' is continuous there.

Then $\int_\gamma f'\,dz = f(B) - f(A)$



<u>Note</u>: Later, we will show that

f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analytic $\Rightarrow \cdots$

$\Rightarrow$ f' continuous

Don't need red stuff.

<u>Pf</u>: $\int_\gamma f'\,dz = \int_a^b \underbrace{f'(z(t))\, z'(t)}_{= \frac{d}{dt} f(z(t))} dt$ Chain rule #2

$= f(z(b)) - f(z(a))$ Fund Thm Calc #1

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Lemma (Basic estimate)

$$\left|\int_\gamma f\,dz\right| \le \left(\max_{z\in tr(\gamma)} |f(z)|\right)\cdot \text{Length}(\gamma)$$

where $\text{Length}(\gamma) = \int_a^b \sqrt{\dot{x}^2+\dot{y}^2}\,dt = \int_a^b |z'(t)|\,dt.$

Pf: $\left|\int_\gamma f\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|$

$$\le \int_a^b \underbrace{|f(z(t))|}_{\le M = \max_\gamma |f|}\,|z'(t)|\,dt$$

$$\le M\underbrace{\int_a^b |z'(t)|\,dt}_{=L} \quad \checkmark$$

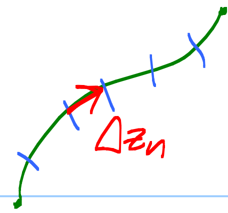
Fact: $\gamma: z(t) \quad a\le t\le b$

$-\gamma: z(b+t(a-b)) \quad 0\le t\le 1$

$$\boxed{\int_{-\gamma} f\,dz = -\int_\gamma f\,dz}$$

Pf: Write out. Equate real and imag parts.
Use real variables change of var formula.

Or: Konrad Knopf

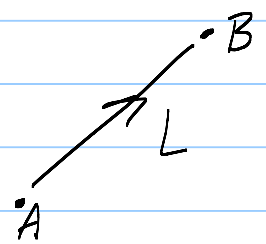


For $-\gamma$, $\Delta z_n = -\Delta z_n$.

Note: Assume $z'(t) \neq 0$ so we don't stop.

Fact: $\int$'s are independent of the parameterization.

EX:

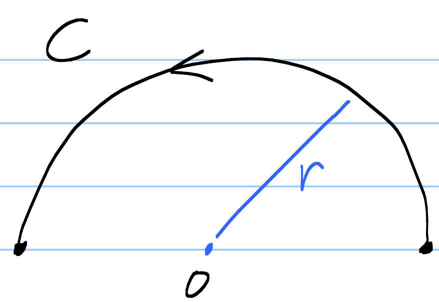


L: $z(t) = A + t(B-A)$

$0 \leq t \leq 1$

$z'(t) = B - A$

EX:



C: $z(t) = re^{it}$

$0 \leq t \leq \pi$

$z'(t) = ire^{it}$

$$\frac{d}{dt} E(it) = \underbrace{E'(it)}_{E(it)} \underbrace{\frac{d}{dt}(it)}_{i}$$

✓

EX: $\displaystyle\int_C z^n \, dz$, $n = -1$

$\frac{d}{dz}\left[\frac{z^{n+1}}{n+1}\right]$

Use Fund Thm Calc #2.