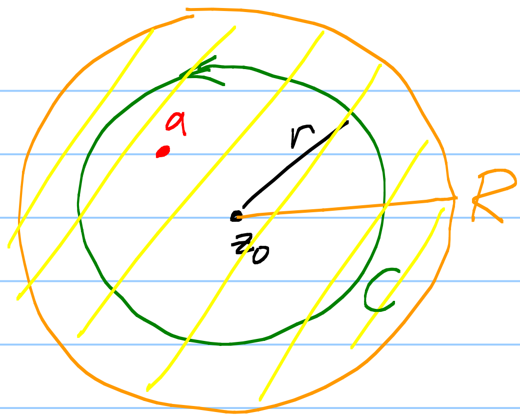
Lecture 8 Liouville's Theorem & the Fundamental Theorem of Algebra

HWK 3 due Thurs, GS



$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz \quad \leftarrow \text{Cauchy Integral Formula}$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz \quad \leftarrow \text{HWK 2: \#1}$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz \quad \leftarrow \text{same pf!}$$

Thm: f analytic on $\Omega \implies f'$ analytic on Ω

$\underset{=u+iv}{\uparrow}$ $\implies u, v$ C^∞-smooth harmonic!

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Cauchy estimates: Let $a = z_0$. Get

$$|f^{(n)}(z_0)| \le \frac{n!\,M}{r^n} \quad \text{where } M = \max_C |f|$$

Pf: $$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\left(\max_{z\in C} \frac{|f(z)|}{\underbrace{|z-z_0|}_{r}{}^{n+1}}\right)\underbrace{\text{Length}(C)}_{2\pi r} \quad \checkmark$$

Case $n=1$: $$|f'(z_0)| \le \frac{M}{r}$$

Liouville's Thm: A $\underbrace{\text{bounded}}_{|f|\le M \text{ on } \mathbb{C}}$ $\underbrace{\text{entire function}}_{\text{analytic on } \mathbb{C}}$ must be constant

Pf: Pick z_0. Cauchy est ($n=1$): $|f'(z_0)| \le \frac{M}{r}$

Let $r \to \infty$. See $f'(z_0) = 0$. z_0 arbitrary.

So $f' \equiv 0$. $\Rightarrow$ $f \equiv$ const. ✓

Way false $\mathbb{R} \to \mathbb{R}$: $\sin x$ [Hmmm. $\sin z$ is not bounded on $\mathbb{C}$.]

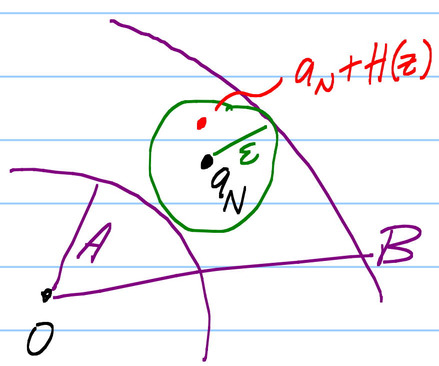
Basic polynomial estimate: $P(z) = a_N z^N + \cdots + a_1 z + a_0$ complex poly of deg $N \ge 1$ ($a_N \ne 0$). There exists a radius $R > 0$ and real constants $0 < A < B$ such that

$$A|z|^N \le |P(z)| \le B|z|^N \text{ when } |z| > R.$$

Remark: $A < |a_N| < B$ $\leftarrow$ A, B can be taken as close to $|a_N|$ as desired.

Pf:

$$\frac{P(z)}{z^N} = a_N + \underbrace{\left(\frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N}\right)}_{H(z)},\quad H(z) \to 0 \text{ as } |z| \to \infty.$$



Take $0 < \varepsilon < |a_N|$.

$|H(z)| < \varepsilon$ if $|z| > R$.

$$A < \left|\frac{P(z)}{z^N}\right| < B \quad \text{if} \quad |z| > R.$$

Books: $\varepsilon = \frac{|a_N|}{2}$.

Fundamental Theorem of Algebra: A complex poly of deg $N \geq 1$ has a root in $\mathbb{C}$. Consequently, can factor polys over $\mathbb{C}$.

Pf: Suppose $P(z)$ does not have a root.

Basic poly est: $A|z|^N \leq |P(z)| \leq B|z|^N$, $|z| > R$.

$\frac{1}{P(z)}$ is an entire fcn!

$$\left|\frac{1}{P(z)}\right| \leq \frac{1}{A|z|^N} \quad \text{when} \quad |z| > R$$

$$< \frac{1}{AR^N} \quad \text{when} \quad |z| > R$$

And: $\left|\frac{1}{P(z)}\right|$ continuous on $\overline{D_R(0)}$ ← closed, bdd, so compact

has max M on $\overline{D_R(0)}$.

$$\left|\frac{1}{P(z)}\right| \leq \operatorname{Max}\left(M, \frac{1}{AR^N}\right) \quad \text{on} \quad \mathbb{C}\ !$$

Liouville's $\Rightarrow \frac{1}{P(z)} \equiv \text{const} \Rightarrow P(z) \equiv \text{const}.$ $\lightning$

$$\left[\ P^N(0) = N!\, a_N \neq 0. \ \lightning\ \right]$$

Morera's theorem: Suppose $f : \Omega \to \mathbb{C}$ is a continuous fcn on open set Ω.

If $\int_\gamma f\,dz = 0$ for every closed γ in Ω,

then f is analytic on Ω.

Note: Enough to know only $\int_Q f\,dz$ for $\overline{\triangle} \subset \Omega$.

Pf: Restrict attention to $D_r(z_0) \subset \Omega$.

Define $F(z) = \int_{L_{z_0}^{z}} f\,dw$ on $D_r(z_0)$.

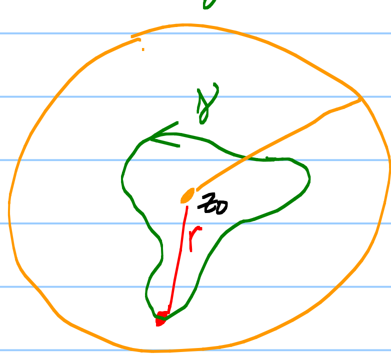
Aha! Our hyp $\Rightarrow$ conclusion of Goursat's!

$\Rightarrow F' = f$. Today: know F analytic

$\Rightarrow F' = f$ analytic.

Cor: Power series converge to analytic fcns inside their circle of convergence.

Pf:



$R = $ R. of C.

$\gamma : z(t),\ a \le t \le b$

$|z(t)|$ continuous on $[a,b]$. It has a max $= r < R$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = \lim_{N\to\infty} \underbrace{P_N(z)}_{\text{polys}}$$

Note $\displaystyle\int_\gamma P_N(z)\,dz = \sum_0^N A_n \int_\gamma z^n\,dz = 0$

$$z^n = \frac{d}{dz}\left[\frac{z^{n+1}}{n+1}\right]$$

Know power series $\to f$ uniformly on $D_r(z_0)$.

$$\left|\int_\gamma f\,dz\right| = \left|\int_\gamma f\,dz - \underbrace{\int_\gamma P_N(z)\,dz}_{=0}\right|$$

$$= \left|\int_\gamma f - P_N\,dz\right| \le \underbrace{\left(\max_\gamma |f-P_N|\right)}_{\substack{\to 0 \text{ as } N\to\infty \\ \text{by unif conv on } \overline{D_r(z_0)}.}} \text{Length}(\gamma)$$

So $\int_\gamma f\,dz = 0$. γ arbitrary. Morera's $\Rightarrow$ f analytic!

Def^n: Given a seq of continuous fcns f_n on an open set $\Omega \subset \mathbb{C}$, we say that f_n converges uniformly on compact subsets of Ω to f if, for each compact $K \subset\subset \Omega$, given $\varepsilon > 0$, there exists an N such that $|f(z) - f_n(z)| < \varepsilon$ for <u>all</u> $z \in K$ when $n > N$. Write $f_n \rightrightarrows f$.

<u>Fact</u>: Uniform limits of continuous are continuous.

So f has to be cont. too.

<u>Thm</u>: f_n analytic and $f_n \rightrightarrows f$, then f is analytic.

<u>Pf</u>: $\Omega =$ Disc. $\left|\int_\gamma f\,dz\right| = \left|\int_\gamma f\,dz - \underbrace{\int_\gamma f_n\,dz}_{=0}\right|$ by Cauchy's

$$= \left|\int_\gamma f - f_n\,dz\right| \le \underset{\substack{tr(\gamma) \\ \text{compact}}}{\text{Max}} |f - f_n| \cdot \text{Length}(\gamma)$$

$$\to 0.$$

Morera's $\Rightarrow$ f analytic.