Lecture 10 Zeroes of analytic functions

HWK 3 due GS Thurs 11:59 pm

Missing piece from Lecture 9: $f(z) = \sum a_n z^n$ ← unif convergence on $D_r(0) \subset D_R(0)$

$$\underbrace{\frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z^{N+1}}\,dz}_{N!\,f^{(N)}(0) = a_N} = \sum a_n \underbrace{\left(\frac{1}{2\pi i}\int_{C_\rho} \frac{1}{z^{1+N-n}}\,dz\right)}_{\substack{=0 \text{ if } N \neq n \\ =1 \text{ if } N = n}}$$

← $0<\rho<r$

$\int\sum = \sum\int$ because of unif conv

Thm: Power series coeff are unique. Taylor's formula.

Lemma If f is analytic on $D_r(z_0)$, $f(z_0) = 0$.

Then, either A) $f \equiv 0$ or

B) zero at z_0 is isolated.

Pf: $f(z) = a_N (z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots$

↑ first $a_n \neq 0$. $(a_0 = f(z_0) = 0)$

$$= (z-z_0)^N \Big[\underbrace{a_N + a_{N+1}(z-z_0) + \cdots}_{F(z)}\Big]$$

Aha! Power series for F converges for exactly same z as p.s. for f!
They have same R.of C. $F(z_0) = a_N \neq 0$.
F analytic on $D_r(z_0) \Rightarrow F$ cont. there. $\exists\ \delta > 0$ such that $F(z) \neq 0$ for all $z \in D_\delta(z_0)$.
z_0 is the only zero of f in $D_\delta(z_0)$ ← "isolated"

Def^n: N is the order (or multiplicity) of the zero of f at z_0.

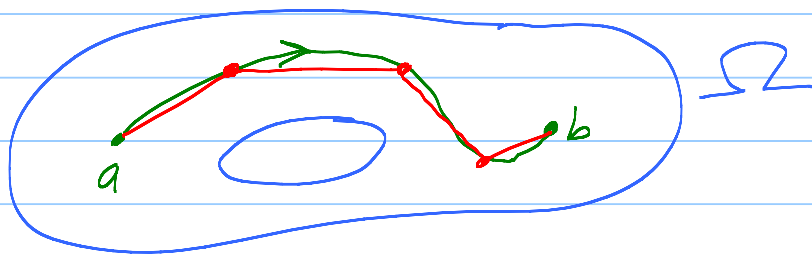
$$f(z) = (z-z_0)^N F(z)$$

where F is analytic where f is and $F(z_0) \neq 0$.

Connected open sets in $\mathbb{C}$ (Domain)

Def^n #1 An open set $\Omega \subset \mathbb{C}$ is path-connected if any two points $a, b \in \Omega$ can be joined by a piecewise C^1 curved γ_a^b in Ω. $(tr(\gamma_a^b) \subset \Omega)$

Topology: γ_a^b is assumed to be merely continuous



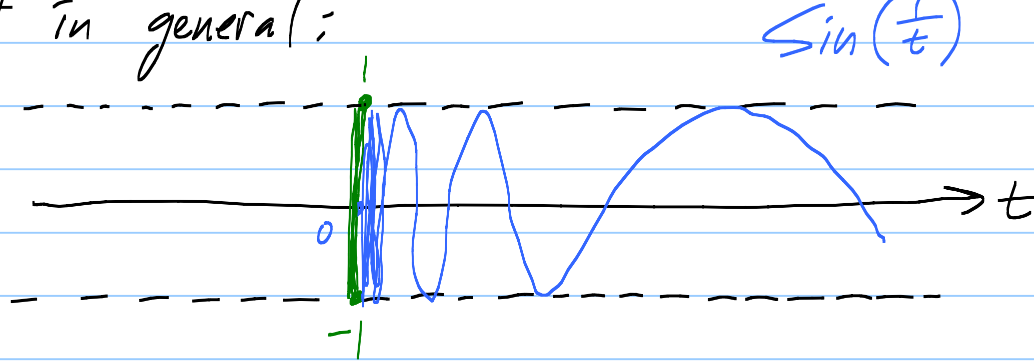
Def^n #2 If U_1 and U_2 are open subsets of Ω (open) and

1) $U_1 \cup U_2 = \Omega$
2) $U_1 \cap U_2 = \emptyset$

then either $U_1 = \Omega$, $U_2 = \emptyset$ or $U_1 = \emptyset$, $U_2 = \Omega$.

Fact Def #1 $\Longleftrightarrow$ Def #2 for open sets Ω.

But in general:



union graph of $\sin\left(\frac{1}{t}\right)$, $t > 0$.

is connected, but not path-connected.

MA 530 : Analytic fcns on <u>domains</u>

<u>Thm</u> f analytic on a domain and $f' \equiv 0$ there, then $f \equiv c$, a const.

<u>Pf</u>: $\displaystyle\int_{\gamma_a^b} \underbrace{f'}_{0}\, dz = f(b) - f(a) = 0$

<u>Thm</u> (Identity theorem) Suppose f is analytic on a domain Ω. Let $Z_f = \{z \in \Omega : f(z) = 0\}$.

Then either A) $f \equiv 0$ $\quad (Z_f = \Omega)$

B) Z_f has <u>no</u> limit points <u>in</u> Ω.

<u>Pf</u>: Suppose $z_0 \in \Omega$ is a limit pt of Z_f, meaning $\exists$ seq $\{z_n\}_{n=1}^{\infty}$ in $Z_f \subset \Omega$, $z_n \neq z_0$, and $z_n \to z_0$ as $n \to \infty$.

<u>Note</u>: $f(z_0) = \lim\limits_{n\to\infty} \underbrace{f(z_n)}_{0} = 0$ by continuity.

So $z_0 \in Z_f$. Z_f is <u>closed</u>.

Ω open. So $\exists\ D_r(z_0) \subset \Omega$, $r > 0$.

Lemma $\Rightarrow$ $f \equiv 0$ on $D_r(z_0)$.

Pick $w_0 \in \Omega$ and let $\gamma: z(t)$, $a \le t \le b$ be curve in Ω connecting z_0 to w_0.

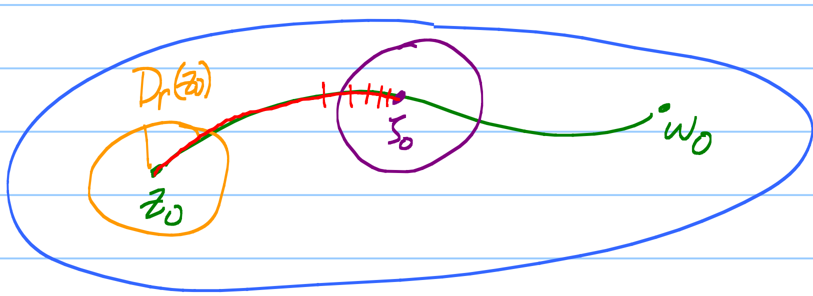
Let $t_{max} = \sup \{T : f(z(t)) \equiv 0,\ t \in [a, T)\}$

$f \equiv 0$ on $D_r(z_0)$ $\Rightarrow$ $t_{max} > a$.

Claim $t_{max} = b$. So $f(w_0) = 0$ too. $f \equiv 0$ on Ω.

Why: Pick seq t_n, $a < t_1 < t_2 < \cdots < t_{max}$ and $t_n \to t_{max}$ as $n \to \infty$. Let $z_n = z(t_n)$.

z_n are zeroes of f that pile up at $\zeta_0 = z(t_{max})$.



Ω open.
$\exists\ D_\rho(\zeta_0) \subset \Omega$
$\rho > 0$.

Lemma $\Rightarrow$ $f \equiv 0$ on $D_\rho(\zeta_0)$. Can get $T > t_{max}$. ↯.

So $t_{max} = b$. ✓

Books: $U_1 = \{z : z \text{ is a limit pt of } Z_f\}$

$U_2 = \Omega - U_1$. Use Def^n #2 if $U_1 \ne \emptyset$.

Consequence e^z is the only analytic extension of e^x from $\mathbb{R}$ to $\mathbb{C}$.

Why: If $f(z)$ and $g(z)$ both extend e^x, then $f-g \equiv 0$ on $\mathbb{R}$. Every pt in $\mathbb{R}$ is a limit pt of $\mathbb{R}$. $\Longrightarrow$ $f-g \equiv 0$ on $\mathbb{C}$.

Cor: Analytic fcns that agree on a subset with a limit pt in a domain agree on the domain.

Consequence: Trig identities extend for comple angles.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

First let $\alpha = z \in \mathbb{C}$. Fix z. Let $\beta = w \in \mathbb{C}$.

Identity Thm way false $\mathbb{R} \to \mathbb{R}$

$$h(t) = \begin{cases} e^{-1/t^2}\sin\left(\frac{1}{t}\right) & t \neq 0 \\ 0 & t = 0 \end{cases}$$

is a C^∞ smooth fcn on $\mathbb{R}$. Zeroes pile up at 0.

All derivatives $=0$ at $t=0$.

$h \neq$ its power series $(\equiv 0)$.

$$f(z) = (z-z_0)^N F(z) = (z-z_0)^N (z-w_0)^M H(z)$$ $H(z_0) \neq 0$, $H(w_0) \neq 0$