Lecture 12 Harmonic functions and conjugates

HWK 4 due Thurs 2/17

Def^n: A harmonic fcn on a domain is a C^2-smooth fcn such that $\Delta u \equiv 0$.

Big tool: If u is harmonic, then

$$f = u_x - i u_y \text{ is analytic.}$$

Think: u has a harmonic conjugate v locally.

$F = u + iv$ is analytic

$$F' = \begin{rcases} \text{Red box \#1} : u_x + i v_x \\ \text{Red box \#2} : v_y - i u_y \end{rcases} = \underbrace{u_x - i u_y}_{\text{analytic}}$$

Pf: $f = u_x - i u_y$ $\quad U = u_x$, $V = -u_y$ $\quad$ u $C^2 \Rightarrow U, V$ C^1

C-R Eqns: $U_x \overset{?}{=} V_y$ Yes! $\Delta u \equiv 0$

$U_y \overset{?}{=} -V_x$ Yes! Mixed partials $= (C^2)$

✓ $U + iV$ is analytic.

Cor: u C^2 harmonic $\Rightarrow$ u is C^∞-smooth!

Cor: If u harmonic on a domain Ω and $\equiv 0$ on $D_\varepsilon(z_0) \subset \Omega$, then $u \equiv 0$ on Ω.

Pf: $f = u_x - i u_y$ is analytic on Ω and $\equiv 0$

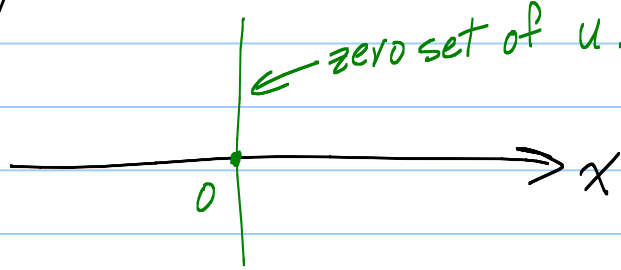
on $D_\varepsilon(z_0) \subset \Omega$. Identity Thm $\Rightarrow$ $f \equiv 0$.

$\Rightarrow$ $\nabla u \equiv 0$ on Ω. $\Rightarrow$ u const on Ω.

$u(z_0) = 0$ $\Rightarrow$ const $= 0$. ✓

Are zeroes of harmonic fcns isolated if not $\equiv 0$? No.

Ex: $u(x,y) = \text{Re}\, z = x$ is harmonic.



Thm: A harmonic fcn on a convex open set Ω has a harmonic conj on Ω.

Pf: Let $f = u_x - i u_y$. [Look at "Think" on p.1]

f analytic. Define $F = U + iV = \int_{L_{z_0}^{z}} f(w)\, dw$.

Recall, $F' = f$

$$\begin{aligned} F' &= U_x + i V_x = U_x - i U_y \\ \| \\ f &\qquad\qquad\quad = u_x - i u_y \end{aligned}$$

Aha! $\nabla(U - u) \equiv 0$ on Ω.

So $U - u \equiv c$, a const on Ω.

Claim: V is a harmonic conjugate for u.

$$u + iV = (U - c) + iV = F - c \quad \text{analytic} \checkmark$$

Cooking up harmonic conjugates on $\mathbb{C}$ or discs.

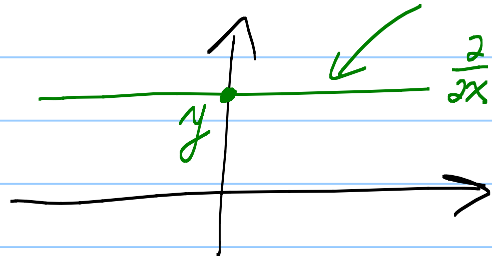
Lemma: $\frac{\partial u}{\partial x} \equiv 0 \iff u$ fcn of y alone

$\frac{\partial u}{\partial y} \equiv 0 \iff u$ fcn of x alone

Pf:

u is const on horizontal lines

$\frac{\partial}{\partial x} u(x,y) \equiv 0$



$$u(x,y) = C(y) \quad (\text{might depend on } y).$$

Conversely: $\frac{\partial}{\partial x} C(y) \equiv 0$.

Cor: $\frac{\partial u_1}{\partial x} \equiv \frac{\partial u_2}{\partial x} \implies u_1 = u_2 + (\text{fcn of } y \text{ alone})$

etc.

Prob: Find a harm conj for

$$u = x^2 - y^2 + x + y \quad \text{on } \mathbb{C} \quad (u \text{ harm} \checkmark)$$

CR-Eqns:

$u_x = v_y \qquad v_x = -u_y = 2y - 1 \quad (A)$

$u_y = -v_x \qquad v_y = u_x = 2x + 1 \quad (B)$

First use (A): $v = \int 2y - 1 \, dx = \boxed{2xy - x + \underline{C(y)}}$

treat y like a const in $\int$

Note: $\frac{\partial}{\partial x}\left(v - (2xy - x)\right) \equiv 0$. They differ by $C(y)$.

Use (B): (In a different way)

(B) $\frac{\partial}{\partial y}\left(\boxed{2xy - x + C(y)}\right) \overset{\text{want}}{=} 2x + 1$

$2x - 0 + C'(y) \overset{\text{want}}{=} 2x + 1$

All x's cancel! $\longrightarrow$ $C'(y) = 1$

$C(y) = y + K$

Finally $v = 2xy - x + \underbrace{(y + K)}_{C(y)}$

Fact: Harmonic conj on a domain is unique up to $+C$.

Pf: If v_1, v_2 both harm for u.

C-R Equs $\Rightarrow$ $\nabla(v_1 - v_2) \equiv 0$

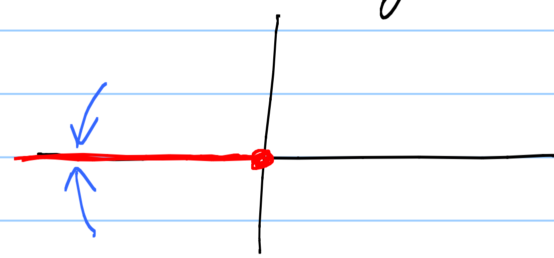
$\Rightarrow$ $v_1 - v_2 \equiv c$,

Danger of holes in Ω

EX: $\ln|z|$ is harmonic on $\mathbb{C}-\{0\}. = \Omega$

Does not have a harm conj on Ω.

Why: HWK 3: $\ln|z| + i \operatorname{Arg} z$ is analytic on $\mathbb{C}-(-\infty,0]$. If v is a harmonic conj for $\ln|z|$ on $\mathbb{C}-\{0\}$, then it is on $\mathbb{C}-(-\infty,0]$ too. So $v = \operatorname{Arg} z + C$ on $\mathbb{C}-(-\infty,0]$.



v has jump discontinuity along cut.

↯.

Complex log fcn:

$w = e^{z} = e^{x+iy} = e^{x} e^{iy}$

$|w| = e^{x}$, $\quad y \in \arg w$

$y = \operatorname{Arg} w + 2\pi n, \quad n \in \mathbb{Z}$

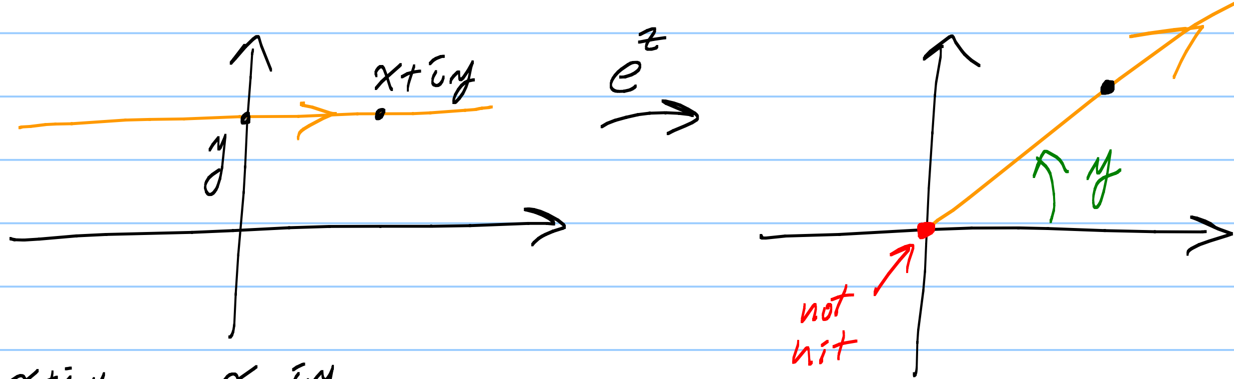
Def^{n}: $\operatorname{Log} w = \ln|w| + i \operatorname{Arg} w$

Principal log $\qquad$ Principal arg

$$\log w = \{ \operatorname{Ln}|w| + i\theta : \theta \in \arg w \}$$

$$= \{ z \in \mathbb{C} : e^z = w \}$$

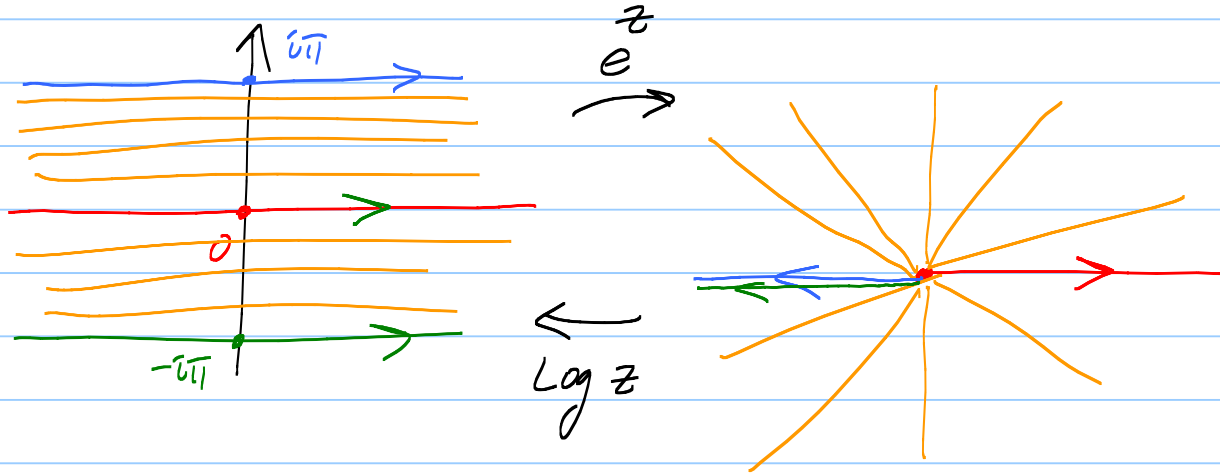
Graph of e^z:



$$e^{x+iy} = e^x e^{iy}$$

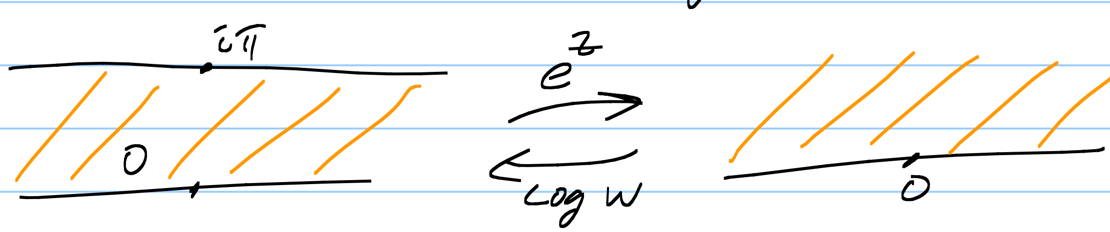
$$-\infty < x < \infty$$

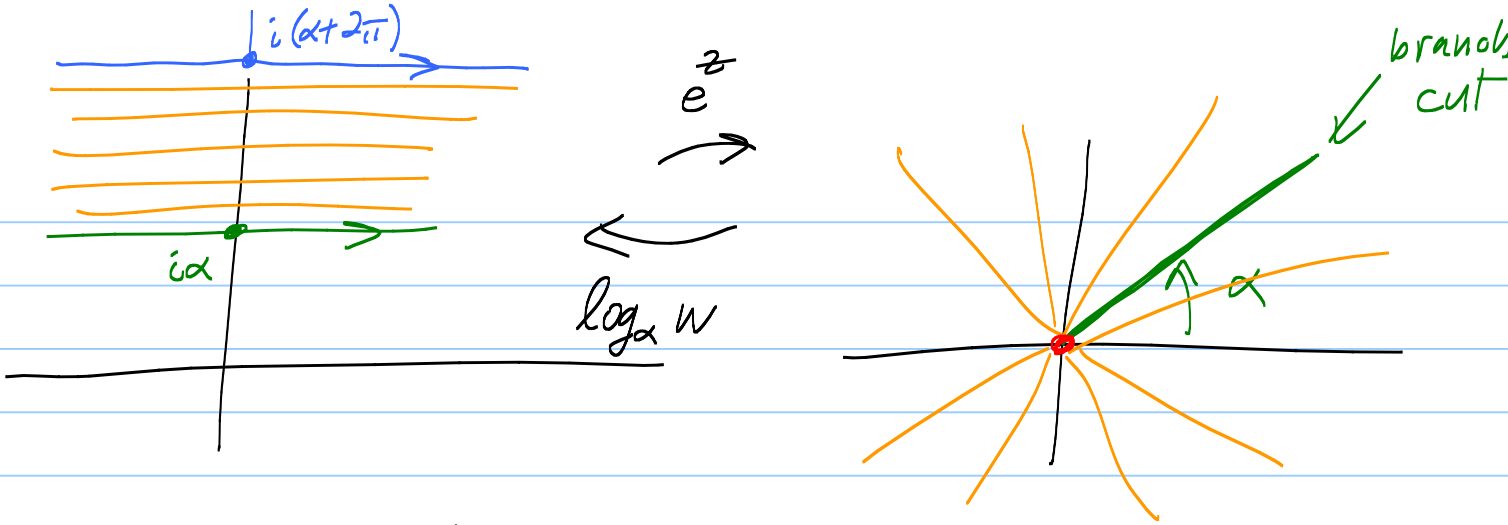
$$0 < e^x < \infty$$



$$e^z : \{ z : -\pi < \operatorname{Im} z < \pi \} \xrightarrow[\text{onto}]{1\text{-}1} \mathbb{C} - (-\infty, 0]$$

$$\xleftarrow[\operatorname{Log} w]{}$$





$\log_\alpha w$ is a "branch" of log fcn.

$$\log_\alpha w = \operatorname{Ln}|w| + i\theta \quad \text{where } \theta \in \arg w$$
$$\alpha < \theta < \alpha + 2\pi$$