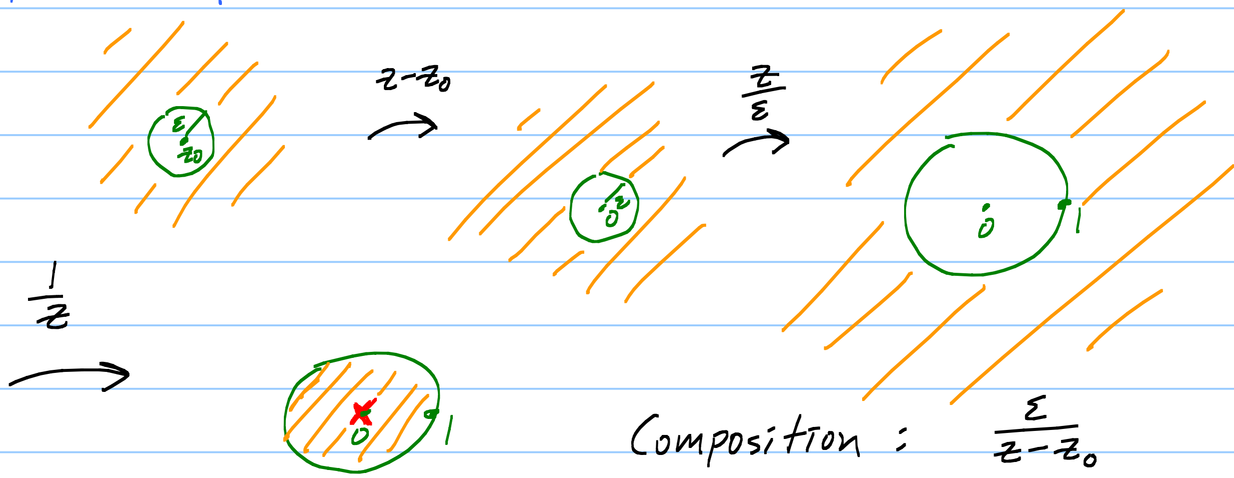
# Lecture 14 Isolated singularities

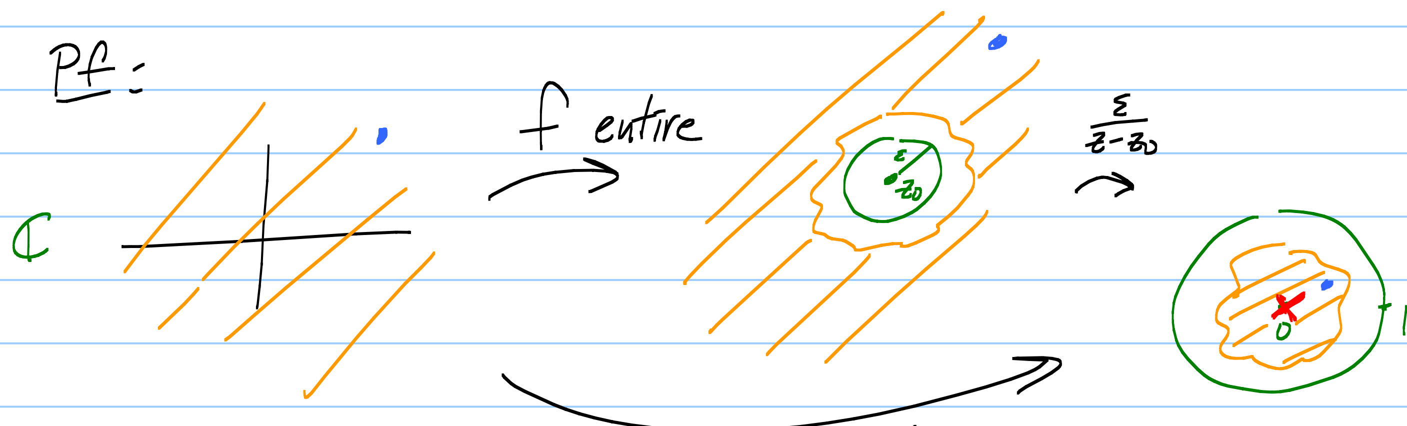
Important map: Outside of small disc to inside of unit disc.




Composition : $\dfrac{\varepsilon}{z - z_0}$

**Casorati-Weierstraß thm** If an entire fcn misses an open set, it must be constant.

Pf:




bdd entire!

Liouville's $\Longrightarrow$ const. $\dfrac{\varepsilon}{f(z) - z_0} = c \neq 0$,

$\Longrightarrow f$ const.

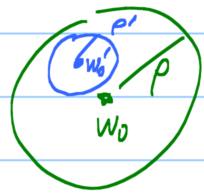
**Thm** Only 3 kinds of isolated singularities.

Pf: Suppose $f$ is analytic on $D_\rho(z_0) - \{z_0\}$.

Suppose $z_0$ is not essential. Then $\exists$ $\varepsilon > 0$ with $0 < \varepsilon < \rho$ such that $\mathcal{A} = f(D_\varepsilon(z_0) - \{z_0\})$ is not dense in $\mathbb{C}$. Then $\exists\, w_0 \in \mathbb{C}$ that $w_0$ is <u>not</u> a limit pt of $\mathcal{A}$.

<u>Case $w_0 \notin \mathcal{A}$</u>: Then $\exists\, \rho > 0$ such that $D_\rho(w_0) \cap \mathcal{A} = \emptyset$.

<u>Case $w_0 \in \mathcal{A}$</u>: Then $\exists\, \rho > 0$ such that $D_\rho(w_0) \cap \mathcal{A} = \{w_0\}$.




Aha! Replace $w_0$ by $w_0'$, $\rho$ by $\rho'$.

Get $\rho > 0$, $w_0 \in \mathbb{C}$ such that $D_\rho(w_0) \cap \mathcal{A} = \emptyset$.


bdd near $z_0$.

R.R.S.T. $\Longrightarrow$ $z_0$ is <u>removable</u> for $h(z) = \dfrac{\rho}{f(z) - w_0}$

Solve for $f$: $\quad f(z) = w_0 + \dfrac{\rho}{h(z)}$

<u>Case</u> $h(z_0) \neq 0$. See $f$ has <u>removable</u> sing at $z_0$.

<u>Case</u> $h(z_0) = 0$. $z_0$ is an <u>isolated</u> zero.

See that $\lim\limits_{z \to z_0} f(z) = \infty$. So $z_0$ is a <u>pole</u>.

<u>Zeroes & poles:</u>

Zeroes: $f(z) = a_N(z-a)^N + \cdots = (z-a)^N \left[ a_N + a_{N+1}(z-a) + \cdots \right]$

$$= (z-a)^N F(z) \quad \text{where } F(a) = a_N = \frac{f^{(N)}(a)}{N!} \neq 0$$

$f$ has a zero of order $N$ at $a$.

Poles: $f(z) = (z-a)^{-N} F(z)$ where $F$ analytic near $a$ and $F(a) \neq 0$

$f$ has a pole of order $N$ at $a$.

Why: If $a$ is a pole of $f$, $\lim_{z \to a} f(z) = \infty$.

So $(M=1)$, $\exists\ \delta > 0$ such that $|f(z)| > 1$

if $z \in D_\delta(a) - \{a\}$. So $f$ nonvanishing there.

$\frac{1}{f}$ has an isolated sing at $a$ and $\left|\frac{1}{f}\right| < 1$

there. R.R.S.T. $\Rightarrow$ $a$ is removable, and

$\lim_{z \to a} \frac{1}{f(z)} = 0 \Rightarrow a$ is an isolated zero

of some finite order $N$. $\frac{1}{f(z)} = (z-a)^N G(z)$

So $f(z) = (z-a)^{-N} F(z)$ where $F = 1/G$. $G(a) \neq 0$,

Def$^n$: $N$ = order of pole at $a$.

Power series routine:

$$f(z) = \frac{1}{(z-a)^N} \Big[ \underbrace{A_0 + A_1(z-a) + A_2(z-a)^2 + \cdots}_{F(z)} \Big]$$

$F(a) = A_0 \neq 0$

residue of f at a

$$= \underbrace{\frac{A_0}{(z-a)^N} + \frac{A_1}{(z-a)^{N-1}} + \cdots + \frac{A_{N-1}}{z-a}}_{R_N(z)} + \underbrace{\left(\begin{matrix}\text{Convergent}\\ \text{power series}\end{matrix}\right)}_{\text{analytic near } a}$$

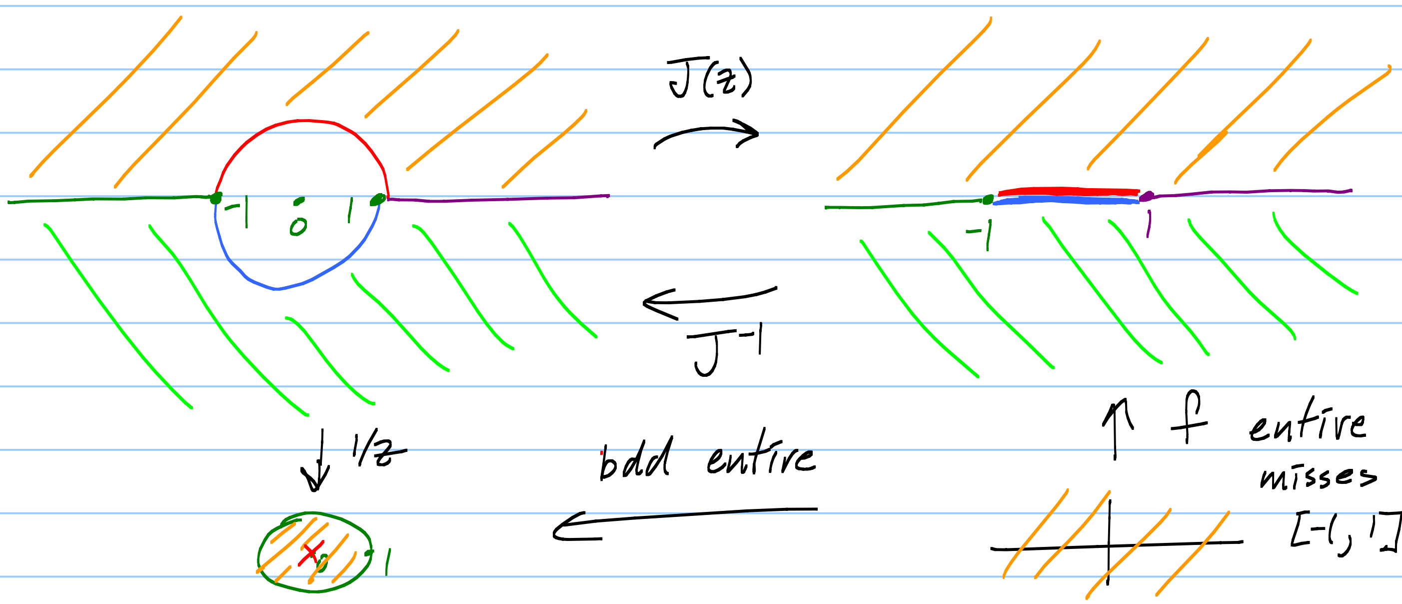
Principal part of f at a

$$f(z) = R_N(z) + h(z)$$ ← analytic

**Fact:** Principal part is uniquely determined by the condition that $f(z) - R_N(z)$ has a removable singularity at a.

**Why:** If $R_1, R_2$ both do this, then $R_1 - R_2$ has a removable sing at a. Show $N_1 = N_2$ and all coeff must be same.

**Famous map** Jukovsky map $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$



Thm: An entire fcn that misses a line segment must be constant.

Picard's little thm Suppose $f$ is entire. Then either, $f$ is a polynomial, or $f$ assumes every value in $\mathbb{C}$ infinitely many times with one possible exception (like $e^z$ does).

Remark: Poly const. $\iff$ has a removable sing at $\infty$.

$P(z)$ poly deg $N>0$ $\iff$ pole at $\infty$.

Not poly $\iff$ $\infty$ is a essential sing.

Def$^n$: $f$ analytic on $\{z: |z|>R\}$.

$$\left(\text{Type of sing of } f \text{ at } \infty\right) \underset{\text{Def}^n}{\overset{\uparrow}{=}} \left(\text{Type of sing of } f\left(\tfrac{1}{z}\right) \text{ at } z=0\right).$$

EX: $e^z$ misses 0. $\cos z$ doesn't miss any value.

Picard's big thm: If $f$ analytic on $D_\rho(z_0)-\{z_0\}$ has an ess sing at $z_0$, $0<\varepsilon<\rho$, then $f$ maps $D_\varepsilon(z_0)-\{z_0\}$ to $\mathbb{C}$ $\infty$-to-one with one possible exception (like $e^{1/z}$ at $z=0$)

Cor of little Picard: An entire fcn that misses two complex values must be constant.

Pf: Polys don't miss anything. Picard ✓